Theorem (*A necessary condition for convergence of a series* )

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem

If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist), then $\sum_{n=1}^{\infty} a_n$ diverges.

The harmonic series is a classic example of a divergent series whose terms limit to zero. For example consider two series - divergent $\sum_{n=1}^{\infty} \frac{1}{n}$ and convergent $\sum_{n=1}^{\infty} \frac{1}{n^2}$ for which limits $\lim_{n\to\infty} \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n^2}$ are 0.

Remark

If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ may or may not converge. In particular, if the sequence does not converge to 0, then the associated series is divergent.

Example

Show that the series $\sum_{n=1}^{\infty} \cos \frac{1}{n}$ diverges.

We have $\lim_{n\to\infty} \cos \frac{1}{n} = \cos 0 = 1 \neq 0$, which implies the series $\sum_{n=1}^{\infty} \cos \frac{1}{n}$ diverges.

Example

Show that the series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

The limit $\lim_{n\to\infty} (-1)^n$ does not exist, which implies the series $\sum_{n=1}^{\infty} (-1)^n$ diverges.