An ordinary differential equation (or ODE) is a relation that contains functions of only one independent variable, and one or more of its derivatives with respect to that variable.
Ordinary differential equations are to be distinguished from partial differential equations where there are several independent variables involving partial derivatives.
Ordinary differential equations arise in many different contexts including geometry, mechanics, astronomy and population modeling.
There are many important classes of differential equations for which detailed information is available.
A function $y = f(x)$ is said to be a solution of a differential equation if the latter is satisfied when y and its derivatives are replaced throughout by $f(x)$ and its corresponding derivatives.

Example

If c_1 and c_2 are arbitrary constants, then

$$y = c_1 \cos x + c_2 \sin x$$

is a solution of the differential equation

$$y'' + y = 0 \quad \left(\frac{d^2y}{dx^2} + y = 0 \right).$$

We will consider:

- First order differential equations:
  - variables separable
  - homogeneous
- Linear first order differential equations
- Linear second order differential equations with constant coefficients

Separable Equations

The differential equation of the form

$$(S) \quad y' = g(x)h(y)$$

is called separable.

In order to solve it, perform the following steps:

(1) Solve the equation $h(y) = 0$, which gives the constant solutions of (S).

(2) Rewrite the equation (S) as

$$\frac{dy}{h(y)} = g(x)dx$$

and, then, integrate to obtain the solutions $(H(y) = G(x) + C)$.

(3) Write down all the solutions - the constant ones obtained from (1) and the ones given in (2).

(4 - optional) Use the initial condition to find the particular solution.

Homogeneous Equations

Occasionally, a differential equation whose variables cannot be separated can be transformed by a change of variable into an equation whose variables can be separated. This is the case with any equation that can be put into the form

$$(H) \quad y' = g\left(\frac{y}{x}\right) .$$

Such an equation is called homogeneous.

To transfer (H) into an equation whose variables may be separated, we introduce the new variable

$$u = \frac{y}{x} .$$

Then

$$y = ux$$

and

$$y' = u'x + u$$

and becomes

$$xu' = g(u) - u .$$

Linear Differential Equations

A first order linear differential equation has the following form:

$$(L) \quad y' + p(x)y = q(x) .$$

The expression

$$u(x) = exp(\int p(x)dx)$$

is called the integrating factor. If an initial condition is given, use it to find the constant C.

Here are some practical steps to follow:

(1) Determine the general solution of the related homogeneous equation $y' + p(x)y = 0$ obtained by replacing $q(x)$ by zero (let y_g be the general solution of this equation).

(2) Find the particular solution of $y' + p(x)y = q(x)$ (let y_p is any particular solution of the nonhomogeneous equation).

(3) Solutions of the nonhomogeneous equation are of the form $y = y_g + y_p$ (the general solution of the nonhomogeneous equation).

(4- optional) Use the initial condition to find C.

We will discuss two ways of finding a particular solution ((2)-y_p):

1. The method of variation of constant
2. Inspired guessing

- If $q(x) = e^{ax}P_n(x)$, then

$$y_s(x) = \begin{cases} e^{ax}Q_n(x) & p \neq -a \\ xe^{ax}Q_n(x) & p = -a \end{cases},$$

  where $P_n(x)$ is a polynomial of degree n, and $Q_n(x)$ is the general form of $P_n(x)$.

- If $q(x) = (k_1 \cos bx + k_2 \sin bx)e^{ax}$, then

$$y_s(x) = (m_1 \cos bx + m_2 \sin bx)e^{ax} ,$$

  where k_1, k_2 are constants, and m_1, m_2 are the general form of constants.

Linear Second-Order Equation with Constant Coefficients

We consider the homogeneous equation

$$ay'' + by' + cy = 0.$$

Theorem

If y_1 and y_2 are two solutions to a linear homogeneous differential equation then so is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \ .$$

The polynomial

$$ar^2 + br + c$$

is called the characteristic polynomial of this equation. Let r_1, r_2 be the roots of characteristic polynomial i.e. solutions of the equation

$$ar^2 + br + c = 0.$$

Case A (Real, distinct roots $r_1 \neq r_2$)

Roots r_1 and r_2 are real and $r_1 \neq r_2$ then equation has two linearly independent solutions

$$y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x} \ .$$

The general solution is

$$y_g(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Case B (Real, double roots $r_1 = r_2$)

Roots r_1 and r_2 are real and $r_1 = r_2 = r$. Then equation has two linearly independent solutions

$$y_1(x) = e^{rx}, y_2(x) = xe^{rx}.$$

The general solution is

$$y_g(x) = C_1 e^{rx} + C_2 x e^{rx}.$$

Case C (Complex roots $r_1 = a + ib$ and $r_2 = a - bi$)

Roots r_1 and r_2 are complex. Then $r_1 = a + ib$ and $r_2 = a - bi$. Equation has two linearly independent solutions

$$y_1(x) = e^{ax}\cos(bx), y_2(x) = e^{ax}\sin(bx) \ .$$

The general solution is

$$y_g(x) = C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx).$$

A linear non-homogeneous ordinary differential equation with constant coefficients has the general form of

$$ay'' + by' + cy = q(x) \ ,$$

where a, b, c are all constants and $q(x) \neq 0$.

For a linear non-homogeneous differential equation, the general solution is the superposition of the particular solution y_p and the complementary solution y_g.

The complementary solution y_g which is the general solution of the associated homogeneous equation ($q(x) = 0$).

Common methodologies on solving the particular solution

- Method of Variation of Parameters

  Steps to follow in applying this method:

  - Find $\{y_1, y_2\}$ a set of fundamental solutions of the associated homogeneous equation and write down the form of the particular solution

  $$y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$$

  - Write down the system

  $$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ q(x) \end{bmatrix}$$

  in other words

  $$\begin{cases} y_1(x)C_1'(x) + y_2(x)C_2'(x) & = & 0 \\ y_1'(x)C_1'(x) + y_2'(x)C_2'(x) & = & q(x) \end{cases}$$

  - Solve it. That is, find $C_1(x)$ and $C_2(x)$ - Plug $C_1(x)$ and $C_2(x)$ into the equation giving the particular solution $y_p(x)$

- Method of Undetermined Coefficient or Guessing Method

  It works only under the following two conditions:

  Condition 1: the associated homogeneous equations has constant coefficients

  Condition 2: the nonhomogeneous term $q(x)$ is a special form

  (a) $q(x) = e^{ax}P_n(x)$ (then $y_p(x) = x^s e^{ax}Q_n(x)$, where s measures how many times r is a root of the characteristic equation) or

  (b) $q(x) = (P_n(x)\cos bx + P_m(x)\sin bx)e^{ax}$ (then $y_p = x^s(Q1_t(x)\cos bx + Q2_t \sin bx)e^{ax}$)

- Method of Reduction of Order

- Method of Inverse Operators

Algorithm for Solving a Second-Order Linear ODE

- Find the general solution, $y_g(x) = C_1 y_1(x) + C_2 y_2(x)$, of the associated homogeneous equation.
- Find any solution, y_p of the inhomogeneous (nonhomogeneous) equation itself. There are two methods available:
  - The Method of Undetermined Coefficients
  - The Method of Variation of Parameters
- Write the general solution

$$y(x) = y_g(x) + y_p(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x).$$

- (optional) Use the initial conditions $y(a) = y_0$, $y'(a) = y_1$ to find C_1, C_2 (these are the two conditions that we'll be using here).

Theorem

If $y_{p1}(x)$ is a particular solution for

$$ay'' + by' + cy = q_1(x)$$

and if $y_{p2}(x)$ is a particular solution for

$$ay'' + by' + cy = q_2(x)$$

then $y_{p1}(x) + y_{p2}(x)$ is a particular solution for

$$ay'' + by' + cy = q_1(x) + q_2(x)\ .$$