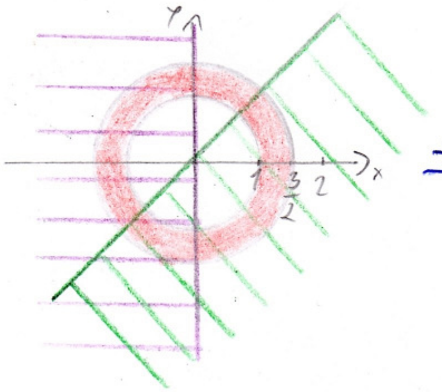
Exercise: Calculate $\iint_D (x^2+y^2+x-1)\,dx\,dy$ where $D = \{(x,y): 1 \le x^2+y^2 \le \frac{9}{4}$ and $x \le 0,\ y \le x\}$

Solution: First, I draw region D

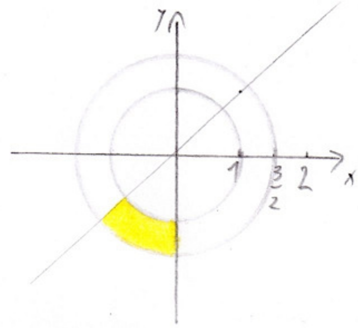
$1 \le x^2+y^2 \le \frac{9}{4}$

$x \le 0$

$y \le x$



$\Rightarrow$

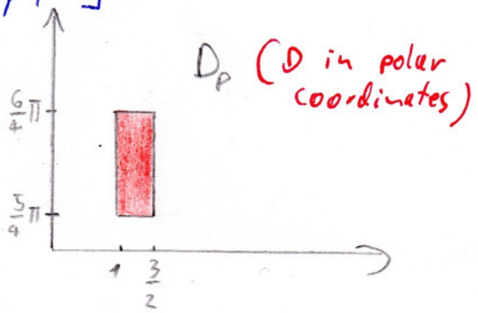


$r \in [1, \frac{3}{2}]$

$\varphi \in [\frac{5}{4}\pi, \frac{6}{4}\pi]$

I will use the formula:

$$\iint f(x,y)\,dx\,dy = \iint_{D_\varphi} f(r\cos\varphi, r\sin\varphi)\cdot \underbrace{r}_{\text{the Jacobian}}\,dr\,d\varphi$$



$f(x,y) = x^2+y^2+x-1$

$$f(r\cos\varphi, r\sin\varphi) = r^2\cos^2\varphi + r^2\sin^2\varphi + r\cos\varphi - 1$$
$$= r(\cos\varphi + r\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}) - 1 = \underline{r\cos\varphi + r^2 - 1}$$

$$\iint_D (x^2+y^2+x-1)\,dx\,dy = \int_{\varphi=\frac{5}{4}\pi}^{\frac{6}{4}\pi}\int_{r=1}^{\frac{3}{2}} (r\cos\varphi + r^2 - 1)\cdot \overset{\text{Jacobian}}{r}\,dr\,d\varphi = \int_{\frac{5}{4}\pi}^{\frac{6}{4}\pi}\left(\int_1^{\frac{3}{2}} (r^2\cos\varphi + r^3 - 1r\,dr)\right)d\varphi = \int_{\frac{5}{4}\pi}^{\frac{6}{4}\pi}\left(\frac{r^3}{3}\cos\varphi + \frac{r^4}{4} - \frac{r^2}{2}\right)\Bigg|_{r=1}^{\frac{3}{2}} d\varphi =$$

$$= \int_{\frac{5}{4}\pi}^{\frac{6}{4}\pi}\left(\frac{27}{24}\cos\varphi + \frac{81}{64} - \frac{9}{8} - \left(\frac{1}{3}\cos\varphi + \frac{1}{4} - \frac{1}{2}\right)\right)d\varphi = \int_{\frac{5}{4}\pi}^{\frac{6}{4}\pi}\left(\frac{27}{24}\cos\varphi + \frac{9}{64} - \frac{1}{3}\cos\varphi + \frac{1}{4}\right)d\varphi =$$

$$= \int_{\frac{5}{4}\pi}^{\frac{6}{4}\pi}\left(\frac{19}{24}\cos\varphi + \frac{25}{64}\right)d\varphi = \frac{19}{24}\sin\varphi + \frac{25}{64}\varphi\Bigg|_{\frac{5}{4}\pi}^{\frac{6}{4}\pi} = \left(-\frac{19}{24} + \frac{25}{64}\cdot\frac{3}{2}\pi\right) - \left(\frac{19}{24}\cdot\left(-\frac{\sqrt{2}}{2}\right) + \frac{25}{64}\cdot\frac{5}{4}\pi\right)$$

$$= -\frac{19}{24} + \frac{75}{128}\pi - \frac{19}{24}\cdot\left(-\frac{\sqrt{2}}{2}\right) - \frac{125}{256}\pi = \frac{25}{256}\pi + \frac{19}{24}\left(-1 + \frac{\sqrt{2}}{2}\right)$$

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