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An example of using polar coordinates in double integrals

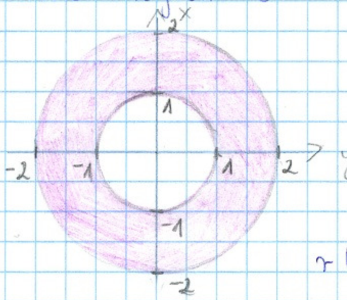
Exercise: Calculate $\iint\limits_D (x^2+y^2+x-1)\,dx\,dy$, where

$$D = \{(x,y):\ 1 \leqslant x^2+y^2 \leqslant 4 \ \text{ and } \ y \geqslant |x|\}$$

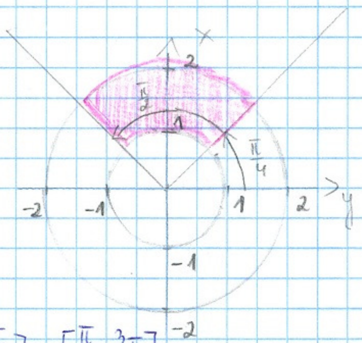
Solution: First, I drew region D

$1 \leqslant x^2+y^2 \leqslant 2^2$

$y \geqslant |x|$

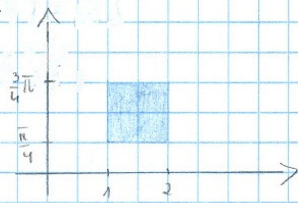


$\Rightarrow$



$r \in [1,2]$

$\varphi \in \left[\frac{\pi}{4}, \frac{\pi}{2}+\frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]$



I will use the formula:

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D_p} f(r\cos\varphi, r\sin\varphi)\cdot \underbrace{r}_{\text{the Jacobian}}\,dr\,d\varphi$$

$f(x,y) = x^2+y^2+x-1$

$$f(r\cos\varphi, r\sin\varphi) = r^2\cos^2\varphi + r^2\sin^2\varphi + r\cos\varphi + 1 = r^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1} + r\cos\varphi + 1 =$$
$$= r^2 + r\cos\varphi + 1$$

$$\iint\limits_D (x^2+y^2+x-1)\,dx\,dy = \int\limits_{\varphi=\frac{\pi}{4}}^{\frac{3}{4}\pi}\int\limits_{r=1}^{2} (r^2+r\cos\varphi-1)\cdot r\,dr\,d\varphi = \int\limits_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\left(\int\limits_1^2 (r^3+r^2\cos\varphi - r)\,dr\right)d\varphi =$$

$$= \int\limits_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\left(\frac{r^4}{4}+\frac{r^3}{3}\cos\varphi - \frac{r^2}{2}\right)\Bigg|_{r=1}^{2} d\varphi = \int\limits_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\left(\frac{16}{4}+\frac{8}{3}\cos\varphi - \frac{4}{2} - \frac{1}{4} - \frac{1}{3}\cos\varphi + \frac{1}{2}\right) =$$

$$= \int\limits_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\left(\frac{7}{3}\cos\varphi + \frac{9}{4}\right)d\varphi = \left(\frac{7}{3}\sin\varphi + \frac{9}{4}\varphi\right)\Bigg|_{\frac{\pi}{4}}^{\frac{3}{4}\pi} = \frac{7}{3}\sin\frac{3}{4}\pi + \frac{9}{4}\cdot\frac{3}{4}\pi - \frac{7}{3}\sin\frac{\pi}{4} - \frac{9}{4}\cdot\frac{\pi}{4} =$$

$$= \frac{7}{3}\cdot\frac{\sqrt{2}}{2} + \frac{18}{16}\pi - \frac{7}{3}\cdot\frac{\sqrt{2}}{2} = \cancel{\frac{7\sqrt{2}}{6}} + \frac{18}{16}\pi - \cancel{\frac{7\sqrt{2}}{6}} = \frac{18}{16}\pi =$$

$$= \underline{\underline{\frac{9}{8}\pi}}$$