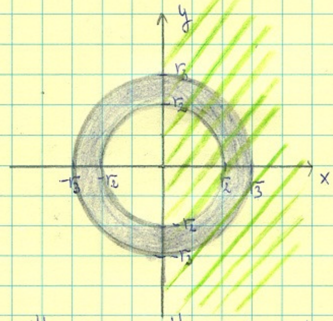
An example of using polar coordinates in double integrals.

Exercise: Calculate $\iint\limits_D \left(\frac{1+x}{x^2+y^2}+3\right)dx\,dy$, where $D=\{(x,y): 2\le x^2+y^2\le 3 \text{ and } x\ge 0\}$

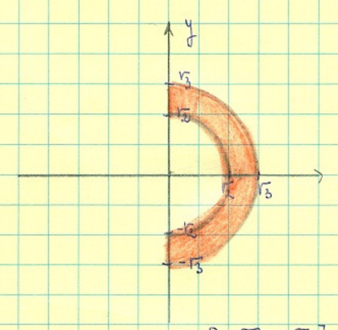
Solution: First, I draw region D:

$2\le x^2+y^2\le 3$

$x\ge 0$



$\Rightarrow$



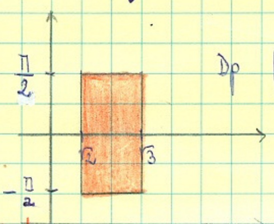
I will use the formula:

$$\iint\limits_D f(x,y)\,dx\,dy=\iint\limits_{D_p} f(r\cos\varphi, r\sin\varphi)\cdot \underset{\substack{\uparrow\\ \text{the Jacobian!}}}{r}\,dr\,d\varphi$$

$r\in[\sqrt{2},\sqrt{3}]$

$\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Downarrow$



D_p (D is polar coordinates)

$$f(x,y)=\left(\frac{1+x}{x^2+y^2}+3\right)$$

$$f(r\cos\varphi, r\sin\varphi)=\frac{1+r\cos\varphi}{r^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}}+3=$$

$$=\frac{1+r\cos\varphi}{r^2}+3$$

$$\iint\limits_D\left(\frac{1+x}{x^2+y^2}+3\right)dx\,dy=\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=\sqrt{2}}^{\sqrt{3}}\left(\frac{1+r\cos\varphi}{r^2}+3\right)\overset{\substack{\text{the Jacobian!}\\ \downarrow}}{r}\,dr\,d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{\sqrt{2}}^{\sqrt{3}}\left(\frac{1}{r}+\cos\varphi+3r\right)dr\right)d\varphi=$$

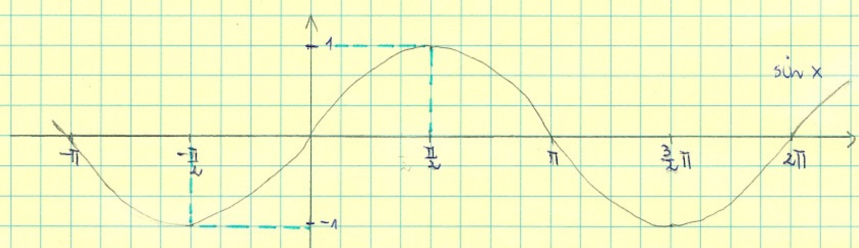
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\left(\ln(r)+r\cos\varphi+3\frac{r^2}{2}\right)\Big|_{\sqrt{2}}^{\sqrt{3}}\right)d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\ln\sqrt{3}+\sqrt{3}\cos\varphi+3\frac{(\sqrt{3})^2}{2}-\ln\sqrt{2}-\sqrt{2}\cos\varphi-3\frac{(\sqrt{2})^2}{2}\right)d\varphi=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\ln\sqrt{\tfrac{3}{2}}+\cos\varphi(\sqrt{3}-\sqrt{2})+4{,}5-3\right)d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\ln\sqrt{\tfrac{3}{2}}+\cos\varphi(\sqrt{3}-\sqrt{2})+1{,}5\right)d\varphi=$$

$$=\left(\varphi\ln\sqrt{\tfrac{3}{2}}+\sin\varphi(\sqrt{3}-\sqrt{2})+1{,}5\varphi\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{\pi}{2}\ln\sqrt{\tfrac{3}{2}}+\sin\left(\tfrac{\pi}{2}\right)(\sqrt{3}-\sqrt{2})+1{,}5\cdot\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\ln\sqrt{\tfrac{3}{2}}-$$

$$-\sin\left(-\tfrac{\pi}{2}\right)(\sqrt{3}-\sqrt{2})-1{,}5\cdot\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}\ln\sqrt{\tfrac{3}{2}}+(\sqrt{3}-\sqrt{2})+0{,}75\pi+\frac{\pi}{2}\ln\sqrt{\tfrac{3}{2}}+(\sqrt{3}-\sqrt{2})+0{,}75\pi$$

$$=\pi\ln\sqrt{\tfrac{3}{2}}+2\sqrt{3}-2\sqrt{2}+1{,}5\pi=\frac{1}{2}\pi\ln\frac{3}{2}+2\sqrt{3}-2\sqrt{2}+1{,}5\pi=\frac{1}{2}\left(-4\sqrt{2}+4\sqrt{3}+\pi\left(3+\ln\frac{3}{2}\right)\right).$$



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