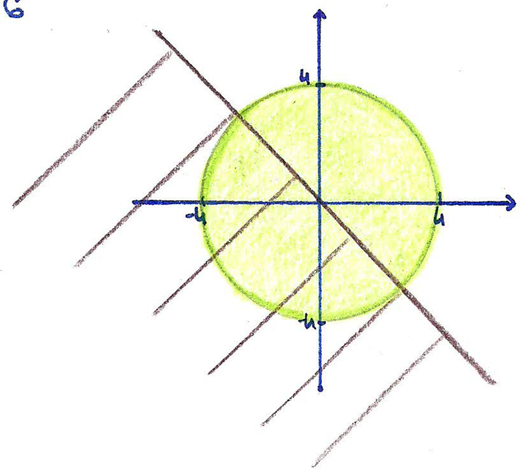
AN EXAMPLE OF USING POLAR COORDINATES IN DOUBLE INTEGRALS

Exercise: Calculate $\iint\limits_D \left(\frac{x}{x^2+y^2}\right) dx\,dy$, where

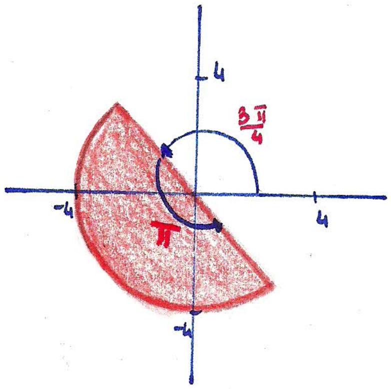
$$D=\{(x,y):\ x^2+y^2 \le 16 \text{ and } y \le -x\}$$

Solution: First, I draw the region D:

- $x^2+y^2 \le 16$
- $y \le -x$

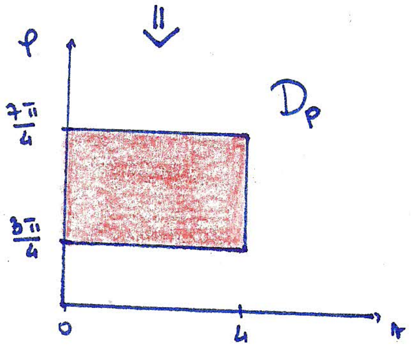


D:



$r \in [0,4]$

$\varphi \in [\frac{3\pi}{4}, \frac{7\pi}{4}]$



I'll use the formula:

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D_\varphi} f(r\cos\varphi, r\sin\varphi)\cdot r\,dr\,d\varphi$$

↑ the Jacobian

$$f(x,y)=\frac{x}{x^2+y^2}$$

$$f(r\cos\varphi, r\sin\varphi)=\frac{r\cos\varphi}{r^2\cos^2\varphi+r^2\sin^2\varphi}=\frac{\cancel{r}\cos\varphi}{r^{\cancel{2}}\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}}=$$

$$=\frac{\cos\varphi}{r}$$

$$\iint\limits_D \left(\frac{x}{x^2+y^2}\right)dx\,dy = \int_{r=0}^{4}\int_{\varphi=\frac{3\pi}{4}}^{\frac{7\pi}{4}}\left(\frac{\cos\varphi}{r}\right)\cdot r\,d\varphi\,dr = \int_{r=0}^{4}\frac{1}{r}\cdot r\,dr\cdot\int_{\varphi=\frac{3\pi}{4}}^{\frac{7\pi}{4}}\cos\varphi\,d\varphi=$$

$$= r\Big|_0^4\cdot\left(-\sin\varphi\Big|_{\frac{3\pi}{4}}^{\frac{7\pi}{4}}\right) = 4-0\cdot\left(-\sin\frac{7\pi}{4}-\left(-\sin\frac{3\pi}{4}\right)\right) =$$

$$= 4\cdot\left(\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\right)=4\cdot\frac{2\sqrt{2}}{2}=4\sqrt{2}$$

KAMILA STANKIEWICZ