We have seen that the geometry of a double integral involves cutting the two-dimensional region into tiny rectangles, multiplying the areas of the rectangles by the value of the function there, adding the areas up, and taking a limit as the size of the rectangles approaches zero. We have also seen that this is equivalent to finding the double iterated integral.

We will now take this idea to the next dimension. Instead of the region in the xy-plane, we will consider a solid in xyz-space. Instead of cutting up the region into rectangles, we will cut up the solid into rectangular solids. And instead of multiplying the function value by the area of the rectangle, we will multipy the function value by the volume of the rectangular solid.

REMARK

Let $f(x, y, z)$ be the density of some three-dimensional solid R. We want to define the triple integral of f over R to be the total mass of R.

We can define the triple integral as the limit of the sum of the product of the function times the volume of the rectangular solids.

If $f(x, y, z)$ is a function defined on a closed bounded region R in space, then we partition a rectangular region about R into rectangular cells by planes parallel to the coordinate planes. The cells have dimension Δx by Δy by Δz. If f is continuous and the bounding surface of R is made of smooth surfaces joined along continuous curves, then

$$\int\int_R\int f(x, y, z)\, dxdydz \overset{def}{=} \lim_{\delta(P)\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)(\Delta x)(\Delta y)(\Delta z),$$

where the point (x_k^*, y_k^*, z_k^*) is arbitrary chosen for $1 \leq k \leq n$.

We call this limit the triple integral of f over R.

REMARK

The limit also exists for some discontinuous functions.

EVALUATION OF TRIPLE INTEGRALS - ITERATED INTEGRALS

Suppose R is a solid region bounded from below by the surface $z = d(x, y)$ and from above by the surface $z = g(x, y)$ that projects onto the region D in the xy-plane. If D is either of type I (vertically simple) or type II (horizontally simple region), then the triple integral of the continuous function over R is equal to

$$\iiint\limits_{R} f(x,y,z)\,dV = \iint\limits_{D}\left(\int\limits_{d(x,y)}^{g(x,y)} f(x,y,z)\,dz\right)dxdy.$$

Let $f(x,y,z)$ be a continuous function over a solid U defined by

$$U = \{(x,y,z): \quad a \le x \le b,\ f(x) \le y \le h(x),\ d(x,y) \le z \le g(x,y)\}\ ,$$

then the triple integral is equal to the triple iterated integral

$$\iiint\limits_{U} f(x,y,z)dV = \int\limits_{a}^{b}\int\limits_{f(x)}^{h(x)}\int\limits_{d(x,y)}^{g(x,y)} f(x,y,z)\,dz\,dy\,dx\ .$$

Theorem

If f and g are both integrable over U, then:

$$\iiint\limits_{U}(f+g)\,dV = \iiint\limits_{U} f\,dV + \iiint\limits_{U} g\,dV\ ;$$

$$\iiint\limits_{U}(f-g)\,dV = \iiint\limits_{U} f\,dV - \iiint\limits_{U} g\,dV\ ;$$

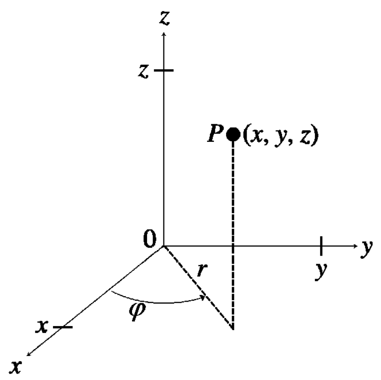
$$\iiint\limits_{U}(c\cdot f)\,dV = c\cdot\iiint\limits_{U} f\,dV, \quad c \in R\ .$$

Triple integrals in Cylindrical and Spherical Coordinates

The cylindrical and spherical coordinate substitutions are special cases of a substitution method that pictures changes of variables in triple integrals as transformations of three-dimensional regions.

Cylindrical Coordinates

The cylindrical coordinate system is a three-dimensional coordinate system which essentially extends circular polar coordinates by adding a third coordinate (usually denoted h) which measures the height of a point above the plane. A point P is given as (r,φ,h).



In terms of the Cartesian coordinate system:

- r is the distance from O to P', the orthogonal projection of the point P onto the xy-plane. This is the same as the distance of P to the z-axis,
- φ is the angle between the positive x-axis and the line OP', measured counterclockwise,
- h is the same as z.

Every point (x, y, z) both Cartesian and cylindrical coordinates (r, φ, h):

$$\begin{cases} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= h \end{cases}.$$

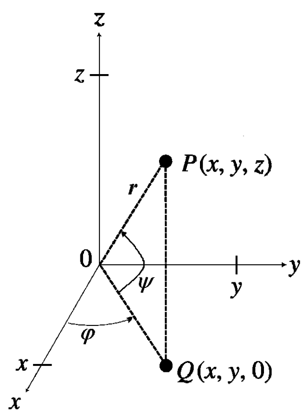
Jacobian of this transformation is $J_T = r$.

Using the change of variable formula, we have the following result for cylindrical coordinates:

$$\iiint\limits_{U} f(x,y,z)dxdydz = \iiint\limits_{\Omega} f(r\cos\varphi, r\sin\varphi, h)\cdot r\; dhdrd\varphi.$$

SPHERICAL COORDINATES

The spherical coordinate system is a coordinate system for representing geometric figures in three dimensions using three coordinates: the radial distance r of a point from a fixed origin, the zenith angle between xy-plane and z-axis, and the azimuth angle from the positive x-axis.



The three coordinates (φ, ψ, r) are defined as:

- $r \geq 0$ is the distance from the origin to a given point P,
- φ ($0 \leq \varphi \leq 2\pi$ or $-\pi \leq \varphi \leq \pi$) is the angle between the positive x-axis and the line from the origin to the P projected onto the xy-plane,
- ψ ($-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$) is the angle between xy-plane and z-axis.

Every point (x, y, z) both Cartesian and spherical coordinates (φ, ψ, r):

$$\begin{cases} x &= r\cos\varphi\cos\psi \\ y &= r\sin\varphi\cos\psi \\ z &= r\sin\psi \end{cases}.$$

Jacobian of this transformation is $J_T = r^2\cos\psi$.

Using the change of variable formula, we have the following result for spherical coordinates:

$$\iiint\limits_{U} f(x,y,z)dxdydz =$$

$$= \iiint\limits_{\Omega} f(r\cos\varphi\cos\psi, r\sin\varphi\cos\psi, r\sin\psi)\cdot r^2\cos\psi\ drd\psi d\varphi.$$

Applications of Triple Integrals

- The volume of a closed bounded region U in space is given by the formula:

$$\text{Volume of } U = \iiint\limits_{U} dxdydz\ .$$

- If $\rho(x,y,z)$ is the density of an object occupying a region R in space, then the integral of ρ over R gives the mass M of the object:

$$M = \iiint\limits_{R} \rho(x,y,z)\ dxdydz\ .$$

- Moment formulas for objects in space:

  First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_{U} x\rho(x,y,z)dxdydz\ , \quad M_{xz} = \iiint\limits_{U} y\rho(x,y,z)dxdydz\ ,$$

$$M_{xy} = \iiint\limits_{U} z\rho(x,y,z)dxdydz$$

  Center of mass:

$$x = \frac{M_{yz}}{M}\ , \quad y = \frac{M_{xz}}{M}\ , \quad z = \frac{M_{xy}}{M}$$