Partial Derivatives - Examples

Let f be a function of two variables. If we fix one of the two variables, say $y = y_0$, the function whose values are $f(x, y_0)$ is a function of x alone. If that function has a derivative at x_0, we call the derivative a partial derivative at (x_0, y_0).

Partial derivatives of f are frequently denoted

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

and

$$f_x, \quad f_y.$$

Definition

Let f be a function of two variables and let (x_0, y_0) be in domain of f.

The partial derivative of f with respect to x at (x_0, y_0) is defined by

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

provided that this limit exists.

The partial derivative of f with respect to y at (x_0, y_0) is defined by

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

provided that this limit exists.

Example 1. Using the definition, calculate first order partial derivatives of $f(x, y) = x\sin(xy)$ at $(x_0, y_0) = (\pi, 1)$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x}(\pi, 1) &\stackrel{def}{=} \lim_{\Delta x \to 0} \frac{f(\pi + \Delta x, 1) - f(\pi, 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\pi + \Delta x)\sin(\pi + \Delta x) - \pi \sin \pi}{\Delta x} \\
&= \lim_{\Delta x \to 0} (\pi + \Delta x) \frac{-\sin \Delta x}{\Delta x} = \pi \cdot (-1) = -\pi,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y}(\pi, 1) &\stackrel{def}{=} \lim_{\Delta y \to 0} \frac{f(\pi, 1 + \Delta y) - f(\pi, 1)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{\pi \sin(\pi(1 + \Delta y)) - \pi \sin \pi}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{-\pi \sin(\pi \Delta y)}{\Delta y} = -\pi^2 \lim_{\Delta y \to 0} \frac{\sin(\pi \Delta y)}{\pi \Delta y} = -\pi^2 \cdot 1 = -\pi^2.
\end{aligned}$$

Example 2. Using derivation formulas, calculate first order partial derivatives of $f(x,y)=x^2+xy^2+y^3$.

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2+xy^2+y^3)=2x+y^2+0=2x+y^2,\\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2+xy^2+y^3)=0+x\cdot 2y+3y^2=2xy+3y^2.\end{aligned}$$

Example 3. Using derivation formulas, calculate first order partial derivatives of $f(x,y)=e^{x^2\sin y}$.

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(e^{x^2\sin y})=e^{x^2\sin y}\cdot 2x\sin y,\\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(e^{x^2\sin y})=e^{x^2\sin y}\cdot x^2\cos y.\end{aligned}$$

Example 4. Using derivation formulas, calculate first order partial derivatives of $f(x,y,z)=x^y+y^z$.

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^y+y^z)=yx^{y-1},\\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^y+y^z)=x^y\ln x+zy^{z-1},\\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}(x^y+y^z)=y^z\ln z.\end{aligned}$$

Second partial derivatives are defined to be partial derivatives of first partial derivatives, and higher derivatives are similarly defined. If both of the first order partial derivatives exist in a neighborhood (x_0,y_0) and they are functions of x and y, then we can differentiate each with respect to x or y:

$$\frac{\partial^2 f}{\partial x^2}=\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right),\quad \frac{\partial^2 f}{\partial x\partial y}=\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right),$$
$$\frac{\partial^2 f}{\partial y\partial x}=\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right),\quad \frac{\partial^2 f}{\partial y^2}=\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right).$$

Partial derivatives involving more than one variable are called mixed partial derivatives.

NOTATION

Pure second partial derivatives: $f_{xx}\equiv\dfrac{\partial^2 f}{\partial x^2}$, $f_{yy}\equiv\dfrac{\partial^2 f}{\partial y^2}$.

Mixed partial derivatives: $f_{xy}\equiv\dfrac{\partial^2 f}{\partial x\partial y}$, $f_{yx}\equiv\dfrac{\partial^2 f}{\partial y\partial x}$.

Example 5. Calculate all second order partial derivatives of $f(x,y) = xy + \frac{x^2}{y^3}$.

Solution: Firstly, we need to calculate first order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy + \frac{x^2}{y^3}) = y + \frac{2x}{y^3}, \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy + \frac{x^2}{y^3}) = x - \frac{3x^2}{y^4}.$$

Now, we are ready to calculate second order partial derivatives:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(y + \frac{2x}{y^3}) = 0 + \frac{2}{y^3} = \frac{2}{y^3}, \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(x - \frac{3x^2}{y^4}) = 1 - \frac{6x}{y^4}, \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(y + \frac{2x}{y^3}) = 1 - \frac{6x}{y^4}, \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(x - \frac{3x^2}{y^4}) = 0 + \frac{12x^2}{y^5} = \frac{12x^2}{y^5}. \end{aligned}$$

Example 6. Calculate $\frac{\partial^5}{\partial x \partial y^4}(xe^{-y})$.

Solution:

$$\begin{aligned} \frac{\partial^5}{\partial x \partial y^4}(xe^{-y}) &= \frac{\partial^4}{\partial x \partial y^3}\Big(\frac{\partial}{\partial y}(xe^{-y})\Big) = \frac{\partial^4}{\partial x \partial y^3}(-xe^{-y}) \\ &= \frac{\partial^3}{\partial x \partial y^2}\Big(\frac{\partial}{\partial y}(-xe^{-y})\Big) = \frac{\partial^3}{\partial x \partial y^2}(xe^{-y}) \\ &= \frac{\partial^2}{\partial x \partial y}\Big(\frac{\partial}{\partial y}(xe^{-y})\Big) = \frac{\partial^2}{\partial x \partial y}(-xe^{-y}) \\ &= \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial y}(-xe^{-y})\Big) = \frac{\partial}{\partial x}(xe^{-y}) = e^{-y}. \end{aligned}$$

Example 7. Check if function $u(x,y,z) = \ln(x^2 + y^2 + z^2)$ satisfies the equation $\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x}$.

Solution:

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial z} &= \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial z}(\ln(x^2 + y^2 + z^2))\Big) = \frac{\partial}{\partial x}(\frac{2z}{x^2 + y^2 + z^2}) = \frac{-4xz}{(x^2 + y^2 + z^2)^2}, \\ \frac{\partial^2 u}{\partial z \partial x} &= \frac{\partial}{\partial z}\Big(\frac{\partial}{\partial x}(\ln(x^2 + y^2 + z^2))\Big) = \frac{\partial}{\partial z}(\frac{2x}{x^2 + y^2 + z^2}) = \frac{-4zx}{(x^2 + y^2 + z^2)^2}. \end{aligned}$$

Yes, function u satisfies the given equation.

APPLICATIONS - TANGENT PLANE

The graph of a function $f(x,y)$ is a surface in $\mathbf{R}^3$ (three dimensional space) and so we can now start thinking of the plane that is "tangent" to the surface at the point.

If $f(x,y)$ and its partial derivatives are all continuous at (x_0, y_0), then we define the tangent plane of the surface at the point $(x_0, y_0, f(x_0, y_0))$:

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \ .$$

Example 8. Write down the equation of a plane tangent to the graph of $f(x,y) = \sqrt{9 - x^2 - y^2}$ at $P = (\sqrt{2}, -\sqrt{3}, 2)$.

Solution: Firstly, we need to calculate partial derivatives at $P_{xy} = (\sqrt{2}, -\sqrt{3})$:

$$\frac{\partial f}{\partial x}(\sqrt{2}, -\sqrt{3}) = \frac{\partial}{\partial x}(\sqrt{9 - x^2 - y^2})|_{(\sqrt{2},-\sqrt{3})} = \frac{-2x}{2\sqrt{9-x^2-y^2}}|_{(\sqrt{2},-\sqrt{3})} = -\frac{\sqrt{2}}{2},$$
$$\frac{\partial f}{\partial y}(\sqrt{2}, -\sqrt{3}) = \frac{\partial}{\partial y}(\sqrt{9 - x^2 - y^2})|_{(\sqrt{2},-\sqrt{3})} = \frac{-2y}{2\sqrt{9-x^2-y^2}}|_{(\sqrt{2},-\sqrt{3})} = \frac{\sqrt{3}}{2}.$$

The equation of a tangent plane is equal to

$$z - 2 = -\frac{\sqrt{2}}{2}(x - \sqrt{2}) + \frac{\sqrt{3}}{2}(y + \sqrt{3})$$

which may be simplified to

$$\sqrt{2}x - \sqrt{3}y + 2x - 9 = 0.$$

TOTAL DIFFERENTIAL

If f is function of two variables, which is differentiable at (x, y) (in the domain of f), then

$$f(x + h, y + h) = f(x, y) + \frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x, y)k\ .$$

The number

$$\frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x, y)k$$

is called the differential (or total differential) of f at (x, y) (with increments h and k) and is denoted

$$df\ .$$

Thus df depends on x, y, h and k. We can write this formula as

$$dy = \frac{\partial f}{\partial x}(x, y)dx + \frac{\partial f}{\partial y}(x, y)dy\ .$$

Example 9. Using the differential of a function calculate the approximated value of $\frac{\arctan 0.9}{\sqrt{4.02}}$.

Solution: We are going to use the following formula:

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y.$$

Let us assume:

$$f(x, y) = \frac{\arctan x}{\sqrt{y}},\ \ (x_0, y_0) = (1, 4),\ \ \Delta x = -0.1,\ \ \Delta y = 0.02.$$

Then, we have:

$$f(1,4) = 0,125\pi\ ,\ \frac{\partial f}{\partial x} = \frac{1}{(1+x^2)\sqrt{y}}\ ,\ \frac{\partial f}{\partial y} = \frac{\arctan x}{-2y\sqrt{y}}.$$

furthermore, $\frac{\partial f}{\partial x}(1,4) = 0.25$ and $\frac{\partial f}{\partial y}(1,4) = -0.015625\pi$. So:

$$\frac{\arctan 0.9}{\sqrt{4.02}} = f(0.9, 4.02) \approx 0.125\pi + 0.25\cdot(-0.1) - 0.015625\pi\cdot 0.02 = 0.1246875\pi - 0.125 \approx 0.366717.$$