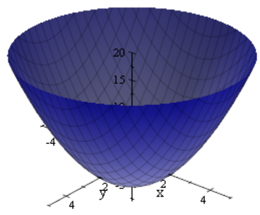
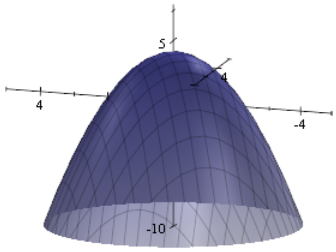
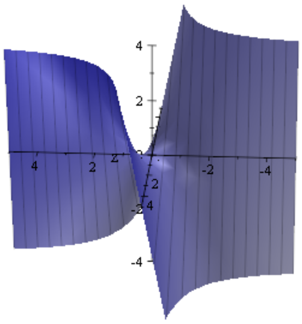
Minima, Maxima and Saddle Points







For a function $f(x,y)$ of two independent variables, we look for points where the surface $z = f(x,y)$ has a horizontal tangent plane. At such points we then look for a local maximum, a local minimum, or a local saddle point.

We organize the search for the local extreme values assumed by a continuous function $f(x,y)$ into two steps:

- The local maxima and minima of f can occur only at points where

$$\frac{\partial f}{\partial x} = 0 \quad \wedge \quad \frac{\partial f}{\partial y} = 0$$

  and points where $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ fail to exist. We call these the critical points of f.

- If f and its first and second partial derivatives are continuous, there is the second derivative test that may identify the behavior of f at critical point (a,b). The expression

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

  is called the discriminant of f and the test goes like this:
  If $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$, then
  i) if $D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, then f has a local maximum at (a,b)
  ii) if $D(a,b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, then f has a local minimum at (a,b)
  iii) if $D(a,b) < 0$, then f has a local saddle point at (a,b)
  iv) if $D(a,b) = 0$, then the test is inconclusive at (a,b) - we must find some other way to determine the behavior of f at (a,b).

MAXIMUM-MINIMUM THEOREM FOR TWO VARIABLES

Let R be a bounded set in the plane that contains its boundary, and let f be a continuous on R. Then f has both a maximum and minimum value on R.

We organize the search for the absolute maximum and minimum values into three steps:

- 1. Find the critical points of f in R, and compute the values of f at this points.
- 2. Find the extreme values of f on the boundary of R.
- 3. The maximum value of f on R will be a largest of the values computed in steps 1. and 2., and the minimum value of f on R will be the smallest of those values.

Example 1. Find local extremes of $f(x, y) = (2x + y^2)e^x$.

Solution: Let us first calculate the first and second order derivatives:

$$\frac{\partial f}{\partial x} = 2e^x + (2x + y^2)e^x = e^x(2 + 2x + y^2), \quad \frac{\partial f}{\partial y} = 2ye^x,$$

$$\frac{\partial^2 f}{\partial x^2} = 2e^x + (2x + y^2 + 2)e^x = e^x(4 + 2x + y^2), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2ye^x, \quad \frac{\partial^2 f}{\partial y^2} = 2e^x.$$

Function f may have extremes only at points, for which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, that is:

$$\frac{\partial f}{\partial x} = e^x(2 + 2x + y^2) = 0, \quad \frac{\partial f}{\partial y} = 2ye^x = 0.$$

Since $e^x \neq 0$ for every x, $2ye^x = 0$ only when $y = 0$. That means:

$$e^x(2 + 2x + y^2) = 0 \Leftrightarrow (2 + 2x + y^2)|_{y=0} = 0 \Leftrightarrow 2 + 2x = 0 \Leftrightarrow x = -1.$$

Function f may have an extreme at $P = (-1, 0)$, so let us calculate the appropriate determinant to make sure:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(-1,0) & \frac{\partial^2 f}{\partial x \partial y}(-1,0) \\ \frac{\partial^2 f}{\partial y \partial x}(-1,0) & \frac{\partial^2 f}{\partial y^2}(-1,0) \end{vmatrix} = \begin{vmatrix} 2e^{-1} & 0 \\ 0 & 2e^{-1} \end{vmatrix} = 4e^{-2} = \frac{4}{e^2} > 0$$

The value of the determinant is positive, therefore f definitely has an extreme at $P = (-1, 0)$. Let us check the value of $\frac{\partial^2 f}{\partial x^2}(-1, 0)$ to check if it's a minimum or a maximum:

$$\frac{\partial^2 f}{\partial x^2}(-1, 0) = 2e^{-1} = \frac{2}{e} > 0,$$

therefore it is a minimum.

Example 2. Find local extremes of $f(x,y) = x\sqrt{y+1} + y\sqrt{x+1}$.

Solution: Let us calculate the first order derivatives:

$$\frac{\partial f}{\partial x} = \sqrt{y+1} + \frac{y}{2\sqrt{x+1}}, \quad \frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y+1}} + \sqrt{x+1}.$$

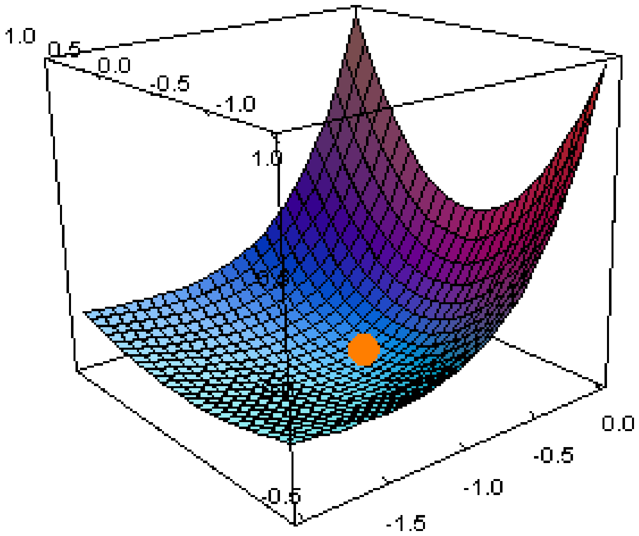
Function f may have an extreme at point $P = (x,y)$ if and only if $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$:

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} \sqrt{y+1} + \frac{y}{2\sqrt{x+1}} = 0 \\ \frac{x}{2\sqrt{y+1}} + \sqrt{x+1} \end{cases} \Leftrightarrow \begin{cases} x = -\frac{2}{3} \\ y = -\frac{2}{3} \end{cases}.$$

Function f may have an extreme only at $P = (-\frac{2}{3}, -\frac{2}{3})$. Let us check the determinant:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(-\frac{2}{3}, -\frac{2}{3}) & \frac{\partial^2 f}{\partial x \partial y}(-\frac{2}{3}, -\frac{2}{3}) \\ \frac{\partial^2 f}{\partial y \partial x}(-\frac{2}{3}, -\frac{2}{3}) & \frac{\partial^2 f}{\partial y^2}(-\frac{2}{3}, -\frac{2}{3}) \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{3}}{2} & \sqrt{3} \\ \sqrt{3} & \frac{\sqrt{3}}{2} \end{vmatrix} = \frac{3}{2} - 9 < 0.$$

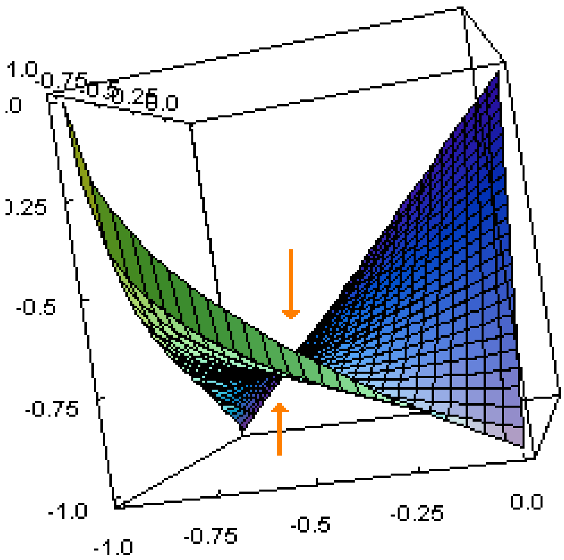
Therefore f does not have an extreme at $P = (-\frac{2}{3}, -\frac{2}{3})$ – it has a saddle point.



Example 1

$f(x,y) = (2x + y^2)e^x$,

$(x,y) \in [-2,0] \times [-1,1]$



Example 2

$f(x,y) = x\sqrt{y+1} + y\sqrt{x+1}$,

$(x,y) \in [-1,0] \times [-1,0]$