Rules of calculating limits of two-variable functions

When calculating a limit of a two-variable function at a point (x_0, y_0), we use the same rules as in case of one-variable functions, **except the de l'Hôspital rule**. There is no de l'Hôspital rule for a function of two variables.

Examples

When the point (x_0, y_0) belongs to the domain of a function, then we can just put the numbers instead of x and y.

Example 1. $\lim\limits_{(x,y)\to(1,2)} \frac{2x-y}{x^2+y^2} = \frac{2\cdot1-2}{1^2+2^2} = \frac{0}{5} = 0.$

Example 2. $\lim\limits_{(x,y)\to(-3,4)} \sqrt{x^2+y^2} = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5.$

When dealing with rational or algebraic functions, it is often useful to multiply the numerator and the denominator by the same expression or to use the short multiplication formulas.

Example 3. $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2+y^2}-1}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \frac{(\sqrt{1+x^2+y^2}-1)(\sqrt{1+x^2+y^2}+1)}{(x^2+y^2)(\sqrt{1+x^2+y^2}+1)} =$
$= \lim\limits_{(x,y)\to(0,0)} \frac{1+x^2+y^2-1}{(x^2+y^2)(\sqrt{1+x^2+y^2}+1)} = \lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{(x^2+y^2)(\sqrt{1+x^2+y^2}+1)} = \lim\limits_{(x,y)\to(0,0)} \frac{1}{\sqrt{1+x^2+y^2}+1} =$
$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$

Example 4. $\lim\limits_{(x,y)\to(1,1)} \frac{x^3-y^3}{y-x} = \lim\limits_{(x,y)\to(1,1)} \frac{(x-y)(x^2+xy+y^2)}{y-x} = \lim\limits_{(x,y)\to(1,1)} -(x^2+xy+y^2) = -3.$

If you come across any trigonometric functions, remember that $\lim\limits_{x\to0} \frac{\sin x}{x} = 1.$

Example 5. $\lim\limits_{(x,y)\to(0,3)} \frac{y^2\sin(x^2)}{x^2} = 3^2\cdot1 = 9.$

Example 6. $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = \lim\limits_{(x,y)\to(0,0)} \frac{(1-\cos(x^2+y^2))(1+\cos(x^2+y^2))}{(x^2+y^2)^2(1+\cos(x^2+y^2))} =$
$= \lim\limits_{(x,y)\to(0,0)} \frac{1-\cos^2(x^2+y^2)}{(x^2+y^2)^2(1+\cos(x^2+y^2))} = \lim\limits_{(x,y)\to(0,0)} \frac{\sin^2(x^2+y^2)}{(x^2+y^2)^2(1+\cos(x^2+y^2))} = \lim\limits_{(x,y)\to(0,0)} \frac{1}{1+\cos(x^2+y^2)} =$
$= \frac{1}{1+1} = \frac{1}{2}.$

There is a similar relation for the natural logarithm: $\lim\limits_{x\to 0} \frac{\ln\frac{1}{x}}{\frac{1}{x}} = 0$.

If the function is similar to $f(x,y)^{g(x,y)}$ then remember, that $\lim\limits_{a(x,y)\to\infty}(1+\frac{1}{a(x,y)})^{a(x,y)} = e$.

Example 7. $\lim\limits_{(x,y)\to(0,0)} \frac{\ln(x^3+1)}{\sin^3(x)} = \lim\limits_{(x,y)\to(0,0)} \frac{\frac{\ln(x^3+1)}{x^3}}{\frac{\sin^3(x)}{x\cdot x\cdot x}} = \lim\limits_{(x,y)\to(0,0)} \frac{\ln(x^3+1)}{x^3}\cdot 1 =$
$= \lim\limits_{(x,y)\to(0,0)} \frac{1}{x^3}\ln(x^3+1) = \lim\limits_{(x,y)\to(0,0)} \ln(1+x^3)^{\frac{1}{x^3}} = \ln\Big(\lim\limits_{(x,y)\to(0,0)}(1+x^3)^{\frac{1}{x^3}}\Big) =$
$= \ln\Big(\lim\limits_{(x,y)\to(0,0)}(1+\frac{1}{\frac{1}{x^3}})^{\frac{1}{x^3}}\Big) = [\frac{1}{x^3}\to\infty \; as \; x\to 0] = \ln e = 1.$

Example 8. $\lim\limits_{(x,y)\to(0,0)}(1+x^2+y^2)^{\frac{1}{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)}(1+\frac{1}{\frac{1}{x^2+y^2}})^{\frac{1}{x^2+y^2}} =$
$= [\frac{1}{x^2+y^2}\to\infty \; as \; (x,y)\to(0,0)] = e.$

Sometimes it is easier to approximate one function by another function using **The Sandwich Theorem**:

THE SANDWICH THEOREM

If

$$\lim_{(x,y)\to(x_0,y_0)} k(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = g$$

and

$$k(x,y)\le f(x,y)\le h(x,y) \text{ for } (x,y) \text{ belonging to a surrounding of } (x_0,y_0)$$

then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = g.$$

Example 9. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = ?$
For every point $(x,y)\neq(0,0)$ we have the following:

$$0\le \frac{x^2y^2}{x^2+y^2} = x^2\frac{y^2}{x^2+y^2} \le x^2\cdot 1 = x^2.$$

Also, $\lim\limits_{(x,y)\to(0,0)} x^2 = 0$. Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.

Some examples may be reduced to a single-variable problem. Then, it is possible to use the de l'Hôspital rule.

Example 10. $\lim\limits_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{\sqrt{x^2+y^2}} = ?$

In this example we can easily substitute the expression $\frac{1}{\sqrt{x^2+y^2}}$ with a new variable u. As (x,y) approaches $(0,0)$, $u = \frac{1}{\sqrt{x^2+y^2}}$ approaches ∞, therefore:

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{\sqrt{x^2+y^2}} = \lim_{u\to\infty} ue^{-u} = \lim_{u\to\infty} \frac{u}{e^u} \overset{H}{=} \lim_{u\to\infty} \frac{1}{e^u} = 0.$$

FUNCTION CONTINUITY

A function $f(x,y)$ is said to be continuous at the point (x_0,y_0) if

- f is defined at (x_0,y_0),
- $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and
- $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

If f and g are both continuous at a point, then their sum $f+g$ is contiuous at that point. Similar results hold for differences, products, and multiples of continuous functions. Also, the quotient of two continuous functions is continuous wherever it is defined.