**Double Integrals - Techniques and Examples**

**Iterated Integrals on a Rectangle**

If function $f$ is continuous on an integral $[a, b] \times [c, d]$, then:

$$\iiint_{[a,b] \times [c,d]} f(x, y) \, dx \, dy = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx.$$

**Notation**

Instead of $\int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy$ we may also write $\int_a^b \int_c^d f(x, y) \, dx \, dy$.

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**Example 1.** Calculate $\iiint_{R} x \, dy \, dx$, where $R = [1, 2] \times [4, 6]$.

**Solution:**

$$\iiint_{R} x \, dy \, dx = \int_a^b \int_c^d x \, dy \, dx = \int_1^2 \left( \int_4^6 x \, dy \right) \, dx = \int_1^2 \left( \int_4^6 x \, dy \right) \, dx = \frac{1}{2} \left( \int_4^6 x \, dy \right) \, dx = \frac{1}{2} \left[ \int_4^6 x \, dy \right] = \frac{1}{2} \left[ \frac{x^2}{2} \right]_4^6 = \frac{1}{2} \left( \frac{36}{2} - \frac{16}{2} \right) = \frac{1}{2} \left( 18 - 8 \right) = \frac{1}{2} \left( 10 \right) = 5.$$

A **double integral of a function with separable variables**

If function $f$ is of form $f(x, y) = g(x) \cdot h(y)$ and $g$ is continuous in $[a, b]$ and $h$ is continuous in $[c, d]$, then:

$$\iiint_{[a,b] \times [c,d]} f(x, y) \, dx \, dy = \int_a^b \left( \int_c^d g(x) \, dx \right) \, dy = \int_c^d \left( \int_a^b h(y) \, dy \right) \, dx.$$

**Example 2.** Calculate $\iiint_{R} x \, dy \, dx$, where $R = [1, 2] \times [4, 6]$, separating variables.

**Solution:**

$$\iiint_{R} x \, dy \, dx = \int_a^b \int_c^d x \cdot \frac{1}{y^2} \, dy \, dx = \left( \int_a^b x \, dx \right) \cdot \left( \int_c^d \frac{dy}{y^2} \right) = \left( \int_a^b x \, dx \right) \cdot \left( \int_c^d \frac{dy}{y^2} \right) = \left( \left[ \frac{x^2}{2} \right]_a^b \right) \cdot \left( \left[ \frac{-1}{y} \right]_c^d \right) = \left( \frac{b^2}{2} - \frac{a^2}{2} \right) \cdot \left( -\frac{1}{d} + \frac{1}{c} \right) = \frac{3}{2} \cdot \frac{1}{12} = \frac{3}{24} = \frac{1}{8}.$$

A **double integral over a simple region**

If $f$ is a continuous function on the vertically simple region

$$D = \{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\},$$

...
then
\[ \int_{D} \int f(x, y) \, dP = \int_{a}^{b} \left( \int_{g(x)}^{h(x)} f(x, y) \, dy \right) \, dx. \]

If \( f \) is a continuous function on the horizontally simple region
\[ D = \{(x, y) : c \leq y \leq d, \ p(y) \leq x \leq q(y)\}, \]
then
\[ \int_{D} \int f(x, y) \, dP = \int_{c}^{d} \left( \int_{p(y)}^{q(y)} f(x, y) \, dx \right) \, dy. \]

**Example 3.** Evaluate \( \int_{D} (x + y) \, dxdy \) over a region bounded by curves \( xy = 6 \) and \( x + y = 7 \). Sketch a diagram of the region.

**Solution:** From the system of equations of \( xy = 6 \) and \( x + y = 7 \) (or: \( y = \frac{6}{x}, \ y = 7 - x \)) we obtain two intersection points: \( A = (1, 6) \) and \( B = (6, 1) \).

Region \( D \) is vertically simple, so:
\[
\int_{D} (x + y) \, dxdy = \int_{1}^{6} \left( \int_{y=\frac{6}{x}}^{7-x} (x + y) \, dy \right) \, dx = \int_{1}^{6} \left( [xy + \frac{y^2}{2}]_{y=\frac{6}{x}}^{7-x} \right) \, dx
\]
\[
= \int_{1}^{6} \left( x(7 - x) + \frac{(7 - x)^2}{2} - x \cdot \frac{6}{x} - \frac{36}{2x^2} \right) \, dx
\]
\[
= \int_{1}^{6} \left( -\frac{x^3}{2} - \frac{18}{x^2} + \frac{37}{2} \right) \, dx = \left[ -\frac{x^3}{6} + \frac{18}{x} + \frac{37x}{2} \right]_{1}^{6} = \frac{125}{3}.
\]

**Example 4.** Evaluate \( \int_{D} (x - y) \, dxdy \) over a region bounded by curves \( x = y^2 \) and \( x = \frac{y^2}{2} + 1 \). Sketch a diagram of the region.

**Solution:** From the system of equations of \( x = y^2 \) and \( x = \frac{y^2}{2} + 1 \) we obtain two intersection points: \( (-\sqrt{2}, 2) \) and \( (\sqrt{2}, 2) \). Region \( D \) is horizontally simple, so:
\[
\int_{D} (x - y) \, dxdy = \int_{-\sqrt{2}}^{\sqrt{2}} \left( \int_{y^2}^{\frac{y^2}{2}+1} (x - y) \, dx \right) \, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left( \left[ \frac{x^2}{2} - xy \right]_{x=y^2}^{x=\frac{y^2}{2}+1} \right) \, dy
\]
\[ \int_{-\sqrt{2}}^{\sqrt{2}} \left( \frac{(\frac{y^2}{2} + 1)^2}{2} - (\frac{y^3}{2} + 1)\frac{y}{2} + \frac{y^4}{2} + y^3 \right) dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left( -\frac{3y^4}{8} + \frac{y^3}{6} - \frac{y^2}{2} - y + \frac{1}{2} \right) dy = \]
\[ = \left[ -\frac{3y^5}{40} + \frac{y^4}{8} + \frac{y^3}{12} - \frac{y^2}{2} + y + \frac{1}{2} \right]_{-\sqrt{2}}^{\sqrt{2}} = \frac{16\sqrt{2}}{15}. \]

**Iterated integrals in a reversed order**

**Example 5.** Sketch the region over which the integration \( \int_{1}^{3} \int_{1-x+2}^{x} (2x + 1) \, dy \, dx \) takes place and write an equivalent integral with the order of integration reversed. Evaluate both integrals.

**Solution:** First let us evaluate:

\[ \int_{1}^{3} \int_{1-x+2}^{x} (2x + 1) \, dy \, dx = \int_{1}^{3} \left( [y(2x + 1)]_{x+2}^{x} \right) dx = \int_{1}^{3} \left( x(2x + 1) - (-x + 2)(2x + 1) \right) dx \]
\[ = \int_{1}^{3} (-2 - 2x + 4x^2) \, dx = \left[ -2x - x^2 + \frac{4x^3}{3} \right]_{1}^{3} = \frac{68}{3}. \]

To reverse the order of integration, we need to divide the region into two parts that are horizontally simple. Now:

\[ \int_{1}^{3} \int_{1-x+2}^{x} (2x + 1) \, dy \, dx = \int_{1}^{3} \int_{y}^{3} (2x + 1) \, dx \, dy + \int_{-1}^{1} \int_{-y+2}^{3} (2x + 1) \, dx \, dy \]
\[ = \int_{1}^{3} \left[ x + x^2 \right]_{y}^{3} \, dy + \int_{1}^{3} \left[ x + x^2 \right]_{y+2}^{3} \, dy = \int_{1}^{3} (12 - y - y^2) \, dy + \int_{-1}^{1} (6 + 5y - y^2) \, dy \]
\[ = \left[ 12y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{1}^{3} + \left[ 6y + \frac{5y^2}{2} - \frac{y^3}{3} \right]_{-1}^{1} = \frac{34}{3} + \frac{34}{3} = \frac{68}{3}. \]
Polar coordinates

For any point \( P \) other than the origin, let \( r \) be the distance between \( P \) and the origin, and \( \varphi \) an angle having its initial side on the positive \( x \) axis and its terminal side on the line segment joining \( P \) and the origin. The pair \((r, \varphi)\) is called a set of polar coordinates for the point \( P \).

Every point \((x, y)\) in the plane has both Cartesian and polar coordinates \((r, \varphi)\):

\[
\begin{align*}
  x &= r \cos \varphi \\
  y &= r \sin \varphi
\end{align*}
\]

We have the following result for polar coordinates:

\[
\int_D \int f(x, y) \, dxdy = \int_{\Delta} \int f(r \cos \varphi, r \sin \varphi) r \, drd\varphi.
\]

**Example 6.** Using polar coordinates, calculate \( \int_D xy^2 \, dxdy \) where \( D : x^2 + y^2 \leq 4, \ x \geq 0 \).

**Solution:** The region of integration is a semicircle with radius equal 2. Therefore, the region in polar coordinates is given by \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) and \(0 \leq r \leq 2\).

After substituting \( x \) and \( y \) with polar coordinates, we have:

\[
\int_D xy^2 \, dxdy = \int_{-\pi/2}^{\pi/2} \left( \int_0^2 (r \cos \theta) \cdot (r \sin \theta)^2 r \, dr \right) d\theta = \int_{-\pi/2}^{\pi/2} \left( \int_0^2 r^4 \sin^2 \theta \cos \theta dr \right) d\theta
\]

\[
= \left( \int_{-\pi/2}^{\pi/2} \sin^2 \theta \cos \theta d\theta \right) \cdot \left( \int_0^2 r^4 dr \right) = \frac{\sin^3 \theta}{3} \bigg|_{-\pi/2}^{\pi/2} \cdot \frac{r^5}{5} \bigg|_0^2 = 64 \frac{15}{15}.
\]

**Example 7.** Using polar coordinates, calculate \( \int_D (x^2 + y^2) \, dxdy \), where \( D : x^2 + y^2 - 2y \leq 0 \).

**Solution (a):** Let us represent the equation describing \( D \) in a different form:

\[
\begin{align*}
  x^2 + y^2 - 2y &\leq 0 \\
  x^2 + (y^2 - 2y + 1) - 1 &\leq 0 \\
  x^2 + (y - 1)^2 &\leq 1
\end{align*}
\]
Such an equation describes a circle with the origin in \((0, 1)\), so we cannot describe it with polar coordinates as easily as in Example 6. Let us substitute \(x = r \cos \theta\) and \(y = r \sin \theta\):

\[
\begin{align*}
x^2 + y^2 - 2y &\leq 0 \\
r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \sin \theta &\leq 0 \\
r &\leq 2 \sin \theta
\end{align*}
\]

the integral is equal to:

\[
\iint_D (x^2 + y^2) \, dx \, dy = \int_0^\pi \left( \int_0^{2 \sin \theta} r^2 (\sin^2 \theta + \cos^2 \theta) \, dr \right) \, d\theta = \int_0^\pi \left( \int_0^{2 \sin \theta} r^3 \, dr \right) d\theta = \int_0^\pi \left( \frac{r^4}{4} \right)_0^{2 \sin \theta} \, d\theta
\]

\[
= 4 \int_0^\pi \sin^4 \theta \, d\theta = 4 \left[ \frac{3\theta}{8} - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} \right]_0^\pi = \frac{3\pi}{2}.
\]

Angle \(\theta\) ranges from 0 to only \(\pi\), because for \(\theta \in (\pi, 2\pi]\) the radius would be negative – which is impossible.

**Solution (b):** Since the circle is moved by a vector of \(\vec{v} = (0, 1)\), then we can also move the function \(x^2 + y^2\) by the same vector. The new function will be \(x^2 + (y - 1)^2\). We can now use the method from Example 6:

\[
\iint_D (x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \left( \int_0^1 \left( r^2 \cos^2 \theta + (r \sin \theta - 1)^2 \right) \, dr \right) d\theta = \cdots = \frac{3\pi}{2}.
\]

**Area of a bounded region in the plane**

The area of a closed bounded plane region \(R\) is given by the formula

\[
\text{Area} = \iint_R 1 \, dx \, dy.
\]

**Example 8.** Calculate the area of a region bounded by curves \(y = \frac{1}{x},\ y = \sqrt{x}\) and a line \(x = 2\). Sketch the region.

**Solution:** The area is equal to:

\[
\int_1^2 \left( \int_{y=\frac{1}{x}}^{\sqrt{x}} \, dy \right) \, dx = \int_1^2 [y]_{y=\frac{1}{x}}^{\sqrt{x}} \, dx = \int_1^2 (\sqrt{x} - \frac{1}{x}) \, dx = \left[ \frac{2}{3} x^{3/2} - \ln |x| \right]_1^2 = \frac{1}{3} (-2 + 4\sqrt{2} - \ln 8).
\]
Volume
Let $R$ be a bounded region in the $OXY$ plane and $f$ be a function continuous on $R$. If $f$ is nonnegative and integrable on $R$, then the volume of the solid region between the graph of $f$ and $R$ is given by

$$\text{Volume} = \iint_R f(x, y) \, dx \, dy.$$ 

Let $R$ be a bounded region in the $xy$ plane and $g_1, g_2$ be continuous functions on $R$. If $g_1$ and $g_2$ are integrable on $R$ such that $g_1(x, y) \leq g_2(x, y)$, then the volume of the solid region between the graph of $g_1$ and $g_2$ is given by

$$\text{Volume} = \iint_R (g_2(x, y) - g_1(x, y)) \, dx \, dy.$$ 

Example 9. Calculate the volume of a solid bounded by curves $y = x^2$, $y = 1$, $z = 0$, $z = 2y$.

Solution: The region of integration is bounded by $y = x^2$ and $y = 1$ and $f(x, y) = 2y$.

Therefore:

$$\text{Volume} = \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} 2y \, dy \, dx = \int_{x=-1}^{x=1} [y^2]_{y=x^2}^{y=1} \, dx = \int_{x=-1}^{x=1} (1 - x^4) \, dx = [x - \frac{x^5}{5}]_{-1}^{1} = 1 - \frac{1}{5} - (-1 + \frac{1}{5}) = 2 - \frac{2}{5} = \frac{8}{5}.$$

Surface
Let $S$ be the surface $z = f(x, y)$ where the points $(x, y)$ come from the given region $R$ in the $OXY$ plane. Then

$$\text{Area}_S = \iint_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} \, dx \, dy,$$

where $f$ and its first partial derivatives are continuous.

Example 10. Calculate the surface of a plane $2x + 2y + z = 8$ bounded by the coordinate system axes.

Solution: After transformations of the equation of a plane, we have $\frac{x}{4} + \frac{y}{4} + \frac{z}{8} = 1$, so the plane intersects the coordinate system axes at points $A = (4, 0, 0)$, $B = (0, 4, 0)$ and $C = (0, 0, 8)$.

Therefore, the region of integration is bounded by $x = 0$, $y = 0$, $y = -x + 4$. We also have
\[ f(x, y) = z = 8 - 2x - 2y, \text{ so } \frac{\partial f}{\partial x} = -2 \text{ and } \frac{\partial f}{\partial y} = -2. \] Therefore:

\[
\text{Surface} = \int_{x=0}^{x=4} \left( \int_{y=0}^{y=-x+4} \sqrt{1 + (-2)^2 + (-2)^2} \, dy \right) \, dx = \int_{x=0}^{x=4} \left( \int_{y=0}^{y=-x+4} \sqrt{9} \, dy \right) \, dx = \int_{x=0}^{x=4} \left( \int_{y=0}^{y=-x+4} 1 \, dy \right) \, dx
\]

\[ = 3 \int_{x=0}^{x=4} \left[ y \right]_{y=0}^{y=-x+4} \, dx = 3 \int_{x=0}^{x=4} (-x + 4) \, dx = 3\left[ -\frac{x^2}{2} + 4x \right]_{0}^{4} = 24. \]