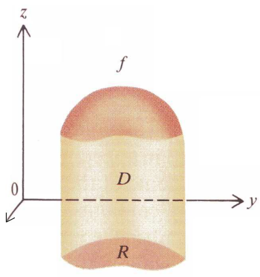
Introduction

From now on we will refer to the definite integral $\int_a^b f(x)dx$ of a function f that is continuous in an interval $[a,b]$ as a single integral. We will define the double integral of a function that is continuous on a certain type of plane region. Consider a region R in the xy plane (do not mistake this notation with the real number set $\mathbf{R}$), a function f that is nonnegative and continuous on R, and the solid region D.



Consider a region R in the xy plane, a function f that is nonnegative and continuous on R, and the solid region D bounded below by R, above by the graph of f, and on the sides by the vertical surface passing through the boundary of R.

We call D the solid region between the graph of f and R.

Our goal is to define the volume of D.

Definition

If for any $\varepsilon > 0$ there is a number $\delta > 0$ such that a partition P of R into n subrectangles whose dimensions are less than δ, then

$$\int\int_R f(x,y)\,dxdy \overset{def}{=} \lim_{\delta(P)\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)(\Delta x)(\Delta y),$$

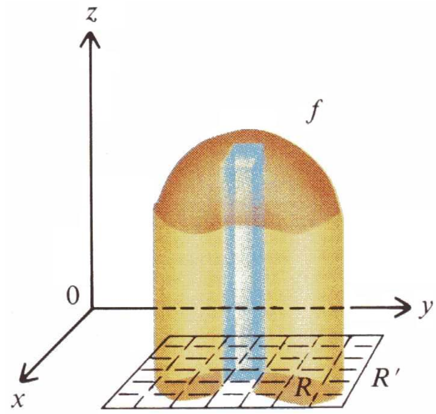
where the point (x_k^*, y_k^*) is arbitrarily chosen for $1 \le k \le n$.

Finally, we relax the assumption that R is a rectangle and assume only that R is bounded and contains its boundary. Then R is contained in a rectangle R'. We partition R' into a collection P of rectangels. In general, some of the rectangles in P will be entirely contained in R, some only partially contained in R, and some will contain no point of R.

Then

$$\int\int_R f(x,y)\,dP \overset{def}{=} \int\int_{R'} f^*(x,y)\,dP,$$

where $f^*(x,y) = \begin{cases} f(x,y) & (x,y)\in R \\ 0 & (x,y)\in R' \end{cases}$.



Definition

Let R be a a bounded region in the xy plane and f a function continuous on R. If f is nonnegative and integrable on R, then the volume V of the solid region between the graph of f and R is given by

$$V = \int\int_R f(x,y)\,dP\,.$$

Definition

A plane region R is vertically simple if there are two continuous functions g_1 and g_2 in the interval $[a,b]$ such that $g_1(x) \le g_2(x)$ for $a \le x \le b$ and such that R is the region between graphs of g_1 and g_2 in $[a,b]$. In this case we say that R is the vertically simple region between the graphs of g_1 and g_2 in $[a,b]$.

A plane region R is horizontally simple if there are two continuous functions h_1 and h_2 in the interval $[c,d]$ such that $h_1(y) \le h_2(y)$ for $c \le y \le d$ and such that R is the region between graphs of h_1 and h_2 in $[c,d]$. In this case we say that R is the horizontally simple region between the graphs of h_1 and h_2 in $[c,d]$.

A plane region R is simple if it is both vertically simple and horizontally simple.

Evaluation of Double Integrals - Iterated Integrals

a) If f is a continuous function on the vertically simple region

$$D = \{(x,y) : a \le x \le b, g(x) \le y \le h(x)\},$$

then

$$\int\int_D f(x,y)\,dP = \int_a^b \Big(\int_{g(x)}^{h(x)} f(x,y)\,dy \Big)\,dx.$$

b) If f is a continuous function on the horizontally simple region

$$D = \{(x, y) : c \le y \le d, p(y) \le x \le q(y)\},$$

then

$$\int\int_D f(x, y)\, dP = \int_c^d \Big(\int_{p(y)}^{q(y)} f(x, y)\, dx \Big)\, dy.$$

THEOREM

Let f and g are integrable on D, then

- $\int\int_D (f(x, y) + g(x, y))dxdy = \int\int_D f(x, y)dxdy + \int\int_D g(x, y)dxdy;$
- $\int\int_D (f(x, y) - g(x, y))dxdy = \int\int_D f(x, y)dxdy - \int\int_D g(x, y)dxdy;$
- $\int\int_D (cf(x, y))dxdy = c \int\int_D f(x, y)dxdy$, where $c \in R$.

EXAMPLE

Sketch the region over which the integration

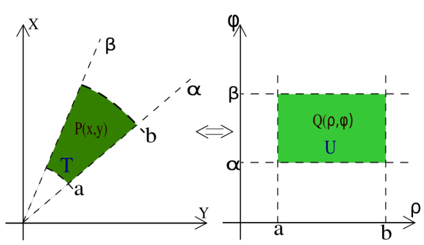
$$\int_0^1 \int_x^{-x+2} (2x + 1)dydx$$

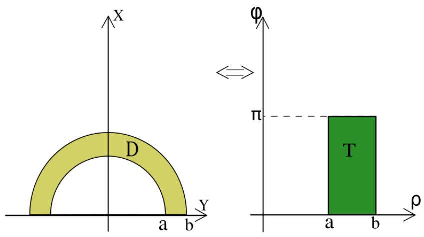
takes place and write an equivalent integral with the order of integration reversed. Evaluate both integrals.

DOUBLE INTEGRALS IN POLAR COORDINATES

For any point P other than the origin, let r be the distance between P and the origin, and φ an angle having its initial side on the positive x axis and its terminal side on the line segment joining P and the origin. The pair (r, φ) is called a set of polar coordinates for the point P. Every point (x, y) in the plane has both Cartesian and polar coordinates (r, φ):

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}.$$





Let Δ and D be regions in uv plane and xy plane. Assume we have a change of variable $x = x(u, v)$ and $y = y(u, v)$. Suppose that the region Δ in the $r\varphi$ - plane is transformed to a region D in the xy - plane under this transformation.

Define the Jacobian of the transformation as

$$J_T(u, v) \stackrel{def}{=} \begin{vmatrix} \frac{\partial \varphi}{\partial u}(u, v) & \frac{\partial \varphi}{\partial v}(u, v) \\ \frac{\partial \psi}{\partial u}(u, v) & \frac{\partial \psi}{\partial v}(u, v) \end{vmatrix}.$$

It turns out that this correctly describes the relationship between the element of area $dxdy$ and the corresponding area element $dudv$.

With this definition, the change of variable formula becomes:

$$\int_D \int f(x, y)\, dxdy = \int_\Delta \int f\left(\varphi(u, v), \psi(u, v)\right) |J_T(u, v)|\, dudv.$$

Note that the formula involves the modulus of the Jacobian.

We first compute the Jacobian for polar coordinates:

$$J_T = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\ .$$

Using the change of variable formula, we have the following result for polar coordinates:

$$\int_D \int f(x, y)\, dxdy = \int_\Delta \int f\left(r\cos\varphi, r\sin\varphi\right) r\, dr d\varphi\ .$$

Polar Coordinates - General Form

$$\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases}$$

where $a > 0$, $b > 0$ and $J_T = abr$.

APPLICATIONS OF DOUBLE INTEGRALS

- Areas of Bounded Regions in the Plane

  The area of a closed bounded plane region R is given by the formula

$$Area = \int\int_R dxdy \ .$$

- Let R be a a bounded region in the xy plane and f be a function continuous on R. If f is nonnegative and integrable on R, then the volume of the solid region between the graph of f and R is given by

$$Volume = \int\int_R f(x,y)\, dxdy \ .$$

  Let R be a a bounded region in the xy plane and g_1, g_2 be continuous functions on R. If g_1 and g_2 are integrable on R such that $g_1(x,y) \leq g_2(x,y)$, then the volume of the solid region between the graph of g_1 and g_2 is given by

$$Volume = \int\int_R [g_2(x,y) - g_1(x,y)]\, dxdy \ .$$

- Another important application of double integrals is is the calculation of surface area.

  Let S be the surface $z = f(x,y)$ where the points (x,y) come from the given region R in the xy plane. Then

$$Area_S = \int\int_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy,$$

  where f and its first partial derivatives are continuous.

- First and Second Moments and Centers of Mass

  Mass and moments formulas for thin plates R covering regions in the xy plane:

  Density:

$$\delta(x,y)$$

  Mass:

$$M = \iint_R \delta(x,y)dxdy$$

  First moments:

$$M_x = \iint_R y\delta(x,y)dxdy \ , \quad M_y = \iint_R x\delta(x,y)dxdy$$

Center of mass:

$$x = \frac{M_y}{M}\ , \quad y = \frac{M_x}{M}$$

Moments of interia (second moments):

About the x-axis:

$$I_x = \iint\limits_R y^2 \delta(x, y) dx dy$$

About the y-axis:

$$I_y = \iint\limits_R x^2 \delta(x, y) dx dy$$

About the origin:

$$I_o = \iint\limits_R (x^2 + y^2) \delta(x, y) dx dy$$

(also called the polar moment of interia about the origin)