Part I – Functions of two variables, their domains and graphs

Exercise 1. Sketch sets of points satisfying the following inequalities.

a) $(x-2)^2+y^2 \le 4$, b) $(x-1)^2+(x+1)^2 < 4$, c) $2x < x^2+y^2$,
d) $x^2+y^2 < 4x$, e) $1 \le x^2+y^2 < 3$, f) $2x < x^2+y^2 < 4x$.

Exercise 2. Sketch sets of points satisfying the following inequalities.

a) $|x| > 2$, b) $|y| \le 3$, c) $|x+y| > 1$, d) $1 \le |x+y| < 4$, .

Exercise 3. Sketch a set of points satisfying $|x^2+y^2-2| \le 1$.

Exercise 4. Sketch sets of points satisfying the following inequalities.

a) $x < 2y+4$, b) $x > 2+y$, c) $y \le 3x+5$,
d) $y \ge -x-1$, e) $y < x^2+2x$, f) $y \ge \sqrt{x}$.

Exercise 5. State the domains of the following functions and draw them in the $\mathbf{R}^2$ plane.

$a(x,y) = \sqrt{x \cdot \sin y}$, $b(x,y) = \arcsin\sqrt{y-\sqrt{x}}$, $c(x,y) = -\sqrt{9-y^2}+x$,
$d(x,y) = \ln(1-x^2-y^2)$, $e(x,y) = \arcsin\frac{x}{y}$, $f(x,y) = \arcsin\frac{y}{x}$,
$g(x,y) = \frac{1}{(x-2)(y+1)}$, $h(x,y) = \frac{1}{x+y}$, $j(x,y) = \ln(x^2+y-2)$.

Exercise 6. Sketch the solid region bounded by:

a) $x^2+y^2=1,\ z=-1,\ z=4$,
b) $z=\sqrt{1-x^2-y^2},\ z=0$,
c) $z=\sqrt{x^2+y^2},\ z=4$,
d) $z=x^2+y^2,\ z=4$,
e) $z^2=x^2+y^2,\ z=-1,\ z=1$,
f) $x^2+y^2=1,\ z=-1,\ z=y+1$,
g) $z=x^2,\ z=1,\ y=4,\ y=-4$,
h) $z=6-x^2-y^2,\ z=\sqrt{x^2+y^2}$,
i) $x+y+z=1,\ x=0,\ y=0,\ z=0$,
j) $x^2-2x+y^2=0,\ z=0,\ z=10$.

Exercise 7. Sketch the graphs of the following functions. Verify your drawings using the 3D function plotter.

$a(x,y) = 2x+4y+8$, $b(x,y) = 3(x^2+(y-2)^2)+1$, $c(x,y) = \sqrt{(x+1)^2+(y+1)^2}-1$,
$d(x,y) = 6-3x-2y$, $e(x,y) = \sqrt{5-x^2-y^2}$, $f(x,y) = 4-x^2-y^2$,
$g(x,y) = y^2-2$, $h(x,y) = 1-\sqrt{2x-x^2+4y-y^2}$, $j(x,y) = 1-\sqrt{(x+2)^2+(y-3)^2}$.

Part II – Limits and continuity

Exercise 8. Draw the first few terms of each of the sequences and find their limits (if they exist) as n approaches infinity.

a) $(x_n, y_n) = (n, \frac{1}{n})$, b) $(x_n, y_n) = (\sqrt[n]{n}, \frac{n}{n+1})$, c) $(x_n, y_n) = ((1+\frac{1}{n})^n, (\frac{2}{3})^n)$,
d) $(x_n, y_n) = ((-1)^n, (-1)^{n+1})$, e) $(x_n, y_n) = (\sin \pi n, \cos \pi n)$, f) $(x_n, y_n) = (2^n, (-1)^n)$,
g) $(x_n, y_n) = (\log_{n+1} 2, \frac{1}{n})$, h) $(x_n, y_n) = (\sqrt[3]{1-\frac{1}{n^3}}, 1)$, i) $(x_n, y_n) = (\arctan n, -\text{arccot } n)$.

Exercise 9. Show that the following limits do not exist. Confirm your result by plotting the function in the neighbourhood of a given point.

a) $\lim\limits_{(x,y)\to(0,0)} \frac{x}{y}$, b) $\lim\limits_{(x,y)\to(0,1)} \frac{x}{y-1}$, c) $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x+y}$,
d) $\lim\limits_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$, e) $\lim\limits_{(x,y)\to(0,1)} \frac{x^6}{y^3-1}$, f) $\lim\limits_{(x,y)\to(0,2)} \frac{y^6}{x^3-8}$,
g) $\lim\limits_{(x,y)\to(0,1)} \frac{x^4}{y^4-1}$, h) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin x}{\sin y}$, i) $\lim\limits_{(x,y)\to(0,\pi)} \frac{\sin x}{\sin y}$.

Exercise 10. Show that limits from **Exercise 9** do not exist by showing different lines of approach.

Exercise 11. Calculate the following limits. Plot each function in the surrounding of the given point.

a) $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2+y^2}-1}{x^2+y^2}$, b) $\lim\limits_{(x,y)\to(1,1)} \frac{x^3-y^3}{y-x}$, c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+3y^3}{\sqrt{x^3+3y^3+1}-1}$,
d) $\lim\limits_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$, e) $\lim\limits_{(x,y)\to(0,3)} \frac{y^2\sin(x^2)}{x^2}$, f) $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$,
g) $\lim\limits_{(x,y)\to(0,3)} \frac{\sin(xy)}{2x}$, h) $\lim\limits_{(x,y)\to(2,0)} \frac{\tan(xy^3)}{y^3}$, i) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+\sin(2xy)}{x}$,
j) $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{x^2y^2}$, k) $\lim\limits_{(x,y)\to(0,0)} (1+x^2+y^2)^{\frac{1}{x^2+y^2}}$, l) $\lim\limits_{(x,y)\to(0,0)} (1+x^2+y^2)^{\frac{3x}{x^2+y^2}}$,
m) $\lim\limits_{(x,y)\to(0,0)} (1+x^4+y^2)^{\frac{4}{x(x^4+y^2)}}$, n) $\lim\limits_{(x,y)\to(0,0)} \frac{\ln(x^3+1)}{\sin^3(x)}$, o) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$,
p) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{2x^2+y^4}$, q) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\cos(x^2+y^2)$, r) $\lim\limits_{(x,y)\to(0,0)} x\sin\frac{1}{x^2+y^2}$.

Hints:

(a) - (d) — use simplified multiplication formulas,

(e) - (g) — use the fact: $\lim\limits_{x\to0} \frac{\sin x}{x} = 1$,

(h) - (n) — use the fact: $\lim\limits_{a(x,y)\to\infty} (1+\frac{1}{a(x,y)})^{a(x,y)} = e$,

(o) - (r) — use the Sandwich Theorem.

Exercise 12. Calculate the following limits (if they exist). If you are not sure, whether the limits exist, first plot the functions in the surrounding of the given point and check if the function graphs behave "suspiciously" at those points.

a) $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$, b) $\lim\limits_{(x,y)\to(0,1)} \frac{x^4}{y^2-1}$, c) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{2x^2+y^4}$,
d) $\lim\limits_{(x,y)\to(1,1)} \frac{x+y-2}{x^2+y^2-2}$, e) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{2x^2+y^2}$, f) $\lim\limits_{(x,y)\to(0,0)} \frac{\ln\frac{x}{y}}{x-y}$.

Exercise 13. Calculate the following limits using various techniques (decide for yourself which technique to use).

a) $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$, b) $\lim\limits_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y}$, c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$,

d) $\lim\limits_{(x,y)\to(0,a)} \frac{y\sin x}{x}$, e) $\lim\limits_{(x,y)\to(0,5)} \frac{\sin(x^2y)}{x^2}$, f) $\lim\limits_{(x,y)\to(3,0)} \frac{\mathrm{tg}(xy)}{y}$,

g) $\lim\limits_{(x,y)\to(\infty,9)} \left(1+\frac{y}{x}\right)^x$, h) $\lim\limits_{(x,y)\to(3,\infty)} \left(1+\frac{1}{xy^2}\right)^{y^2}$, i) $\lim\limits_{(x,y)\to(\infty,7)} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$,

j) $\lim\limits_{(x,y)\to(0,0)} (1+x^2+y^2)^{\frac{1}{x^2+y^2}}$, k) $\lim\limits_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2}$, l) $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$.

Exercise 14. Using polar coordinates, calculate:

- four different examples from **Exercise 9**,
- six different examples from **Exercise 11**,
- two different examples from **Exercise 12**,
- four different examples from **Exercise 13**.

Exercise 15. Check the continuity of the following functions. If a function is discontinuous, suggest a way to change the function so that it becomes continuous.

$$a(x,y) = \begin{cases} 5-x-y & (x,y)\neq(1,2) \\ 1 & (x,y)=(1,2) \end{cases}, \quad b(x,y) = \begin{cases} \frac{sin(2xy)}{x} & x\neq 0, y\in\mathbf{R} \\ 1 & x=0, y\in\mathbf{R} \end{cases},$$

$$c(x,y) = \begin{cases} \frac{x^4-y^4}{x^4+y^4} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}, \quad d(x,y) = \begin{cases} \frac{y^2}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases},$$

$$e(x,y) = \begin{cases} 1-\sqrt{x^2+y^2} & x^2+y^2<1 \\ x^2+y^2-1 & x^2+y^2\geq 1 \end{cases}, \quad f(x,y) = \begin{cases} x+y & x>0, y\in\mathbf{R} \\ \sqrt{x^2+y^2} & x\leq 0, y\in\mathbf{R} \end{cases}.$$

Some exercises were taken from the website of dr Jolanta Dymkowska.