ELEMENTARY MATHEMATICS

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Chapter 7

PROGRESSIONS

7.1. Arithmetic Progressions

EXAMPLE 7.1.1. Consider the finite sequence of numbers

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.$$

This sequence has the property that the difference between successive terms is constant and equal to 2. It follows that the k-th term is obtained from the first term by adding $(k-1) \times 2$, and is therefore equal to $1 + 2(k-1)$. On the other hand, if we want to add all the numbers together, then we observe that

$$1 + 31 = 3 + 29 = 5 + 27 = 7 + 25 = 9 + 23 = 11 + 21 = 13 + 19 = 15 + 17,$$

so that the numbers can be paired off in such a way that the sum of the pair is always the same and equal to 32. Note now that there are 16 numbers which form 8 pairs. It follows that

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 = 8 \times 32 = 256.$$

EXAMPLE 7.1.2. Consider the finite sequence of numbers

$$2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.$$

This sequence has the property that the difference between successive terms is constant and equal to 3. If we want to add all the numbers together, then we observe that

$$2 + 32 = 5 + 29 = 8 + 26 = 11 + 23 = 14 + 20 = 2 \times 17,$$

† This chapter was written at Macquarie University in 1999.

so that the numbers other than the middle one can be paired off in such a way that the sum of the pair is always the same and equal to 34. Note now that there are 11 numbers which form 5 pairs, as well as the number 17 which is equal to half the sum of a pair. We can therefore pretend that there are $5\frac{1}{2}$ pairs, each adding to 34. It follows that

$$2+5+8+11+14+17+20+23+26+29+32 = \frac{11}{2} \times 34 = 187.$$

Definition. By an arithmetic progression of m terms, we mean a finite sequence of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(m-1)d. \tag{1}$$

The real number a is called the first term of the arithmetic progression, and the real number d is called the difference of the arithmetic progression. The term $a+(k-1)d$ is called the k-th term of the arithmetic progression.

SUM OF AN ARITHMETIC PROGRESSION. *The sum of the m terms of an arithmetic progression of the type* (1) *is equal to*

$$\frac{m}{2} \times (2a+(m-1)d).$$

Remark. Note that the sum of an arithmetic progression is equal to

$$\frac{\text{number of terms}}{2} \times (\text{first term} + \text{last term}).$$

Example 7.1.3. Suppose that the 4-th and 7-th terms of an arithmetic progression are equal to 9 and -15 respectively. Then we have

$$\begin{aligned} 9 &= a+3d, \\ -15 &= a+6d, \end{aligned}$$

so that $3d = -24$. It follows that $d = -8$ and $a = 33$. The arithmetic progression is given by

$$33, 25, 17, 9, 1, -7, -15, \ldots.$$

The 10-th term is given by $a+9d = 33-72 = -39$. The sum of the first 10 terms is equal to

$$\frac{10}{2} \times (33-39) = -30.$$

Example 7.1.4. We have

$$1+3+5+\ldots+(2n-1) = \frac{n}{2} \times (1+2n-1) = n^2$$

and

$$2+4+6+\ldots+2n = \frac{n}{2} \times (2+2n) = n+n^2.$$

Note also that

$$1+2+3+\ldots+2n = \frac{2n}{2} \times (1+2n) = n+2n^2.$$

7.2. Geometric Progressions

EXAMPLE 7.2.1. Consider the finite sequence of numbers

$$4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192.$$

This sequence has the property that the ratio between successive terms is constant and equal to 2. It follows that the k-th term is obtained from the first term by multiplying 2^{k-1}, and is therefore equal to $4 \times 2^{k-1}$.

DEFINITION. By a geometric progression of m terms, we mean a finite sequence of the form

$$a, ar, ar^2, ar^3, \ldots, ar^{m-1}. \tag{2}$$

The real number a is called the first term of the geometric progression, and the real number r is called the ratio of the geometric progression. The term ar^{k-1} is called the k-th term of the geometric progression.

Suppose now that we wish to add the numbers in (2). Write

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{m-1}.$$

Then

$$rS = \qquad ar + ar^2 + ar^3 + \ldots + ar^{m-1} + ar^m.$$

It follows that

$$S - rS = a \qquad\qquad\qquad\qquad - ar^m.$$

Hence

$$S = \frac{a - ar^m}{1 - r},$$

provided that $r \neq 1$. On the other hand, if $r = 1$, then $S = am$.

We have proved the following result.

SUM OF A GEOMETRIC PROGRESSION. *The sum of the m terms of a geometric progression of the type* (2) *is equal to am if $r = 1$, and equal to*

$$\frac{a - ar^m}{1 - r}$$

if $r \neq 1$.

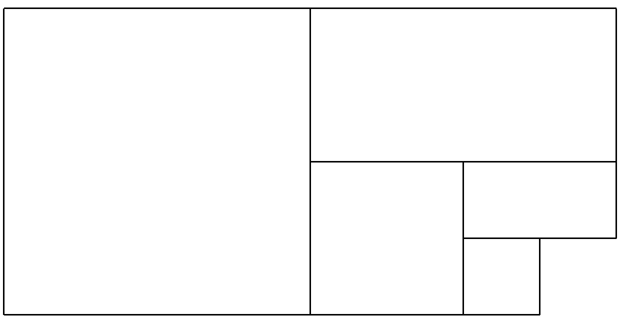
EXAMPLE 7.2.2. Consider the geometric sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$$

The sum of the first m terms is equal to

$$\frac{1 - 2^{-m}}{1 - 2^{-1}} = 2 - \frac{1}{2^{m-1}},$$

very close to 2 when m is very large. We can explain this geometrically by the picture below:



Suppose that the square on the left has area 1. Then the rectangle on the top right has area 1/2. The square the next size down has area 1/4. The rectangle the next size down has area 1/8. The square the next size down has area 1/16. The next term 1/32 will fill half of the missing square on the bottom right. The term 1/64 will fill half of what is still missing on the bottom right. If m is very large, then we account for nearly all of the missing piece and so get a big rectangle of area 2.

Problems for Chapter 7

1. Find each of the following sums without using your calculators, taking care to explain each step of your argument:
   a) $1+3+5+7+\ldots+999$ (sum of arithmetic progression).
   b) $2+4+8+16+\ldots+2^n$ (sum of geometric progression), where $n \in \mathbb{N}$.

2. Using the idea of arithmetic progressions and geometric progressions, without the help of calculators to find the values of the individual terms or to add them together, find the sum

$$5+12+21+\ldots+1048674$$

   of 20 terms, where the k-th term of the sum is given by $2^k+3+5(k-1)$.
   [Hint: Consider the sum of the terms 2^k separately from the sum of the terms $3+5(k-1)$.]

3. Using the idea of arithmetic progressions and geometric progressions, without the help of calculators to find the values of the individual terms or to add them together, find the sum

$$3+10+25+\ldots+39394$$

   of 10 terms, where the k-th term of the sum is given by $2\times 3^{k-1}+1+3(k-1)$.

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