- Linear functions:

The term linear function can refer to either of two different but related concepts:

1. The term linear function is often used to mean a first degree polynomial function of one variable. These functions are called "linear" because they are precisely the functions whose graph in the Cartesian coordinate plane is a straight line.

2. In advanced mathematics, a linear function often means a function that is a linear map, that is, a map between two vector spaces that preserves vector addition and scalar multiplication.

We will consider only first one of above meanings.

Equations of lines:

1. $y = ax + b$ - Slope-intercept equation

2. $y - y_1 = a(x - x_1)$ - Point-slope equation

3. $y = b$ - Horizontal line

4. $x = c$ - Vertical line

where $x \in R$, $a, b, c \in R$ and a is the slope.

If the slope $a \neq 0$, then x-intercept is equal $\dfrac{-b}{a}$.

- Quadratic function:

$f(x) = ax^2 + bx + c$ for $x \in R$, where $a, b, c \in R$, $a \neq 0$.

$$\Delta = b^2 - 4ac$$

This formula is called the discriminant of the quadratic equation.

The graph of such a function is a parabola with vertex $W\left(\dfrac{-b}{2a}, \dfrac{-\Delta}{4a}\right)$.

A quadratic function is also referred to as:

- a degree 2 polynomial,

- a 2nd degree polynomial,

because the highest exponent of x is 2.

If the quadratic function is set equal to zero, then the result is a quadratic equation.

The solutions to the equation are called the roots of the equation or the zeros of the function.

- If $\Delta > 0$, then there are two x-intercepts because the two real roots are distinct $x_1 = \dfrac{-b - \sqrt{\Delta}}{2a}$, $x_1 = \dfrac{-b + \sqrt{\Delta}}{2a}$,

- If $\Delta = 0$, then there is exactly one x-intercept because of the two real roots are equal $x_0 = \frac{-b}{2a}\ (= x_1 = x_2)$,
- If $\Delta < 0$ the graph has no x-intercepts.

Forms of a quadratic function

- the general form or polynomial form: $y = ax^2 + bx + c$, where $a \neq 0$,
- the standard form or vertex form: $y = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$,
- the factored form (if $\Delta \geq 0$): $y = a(x - x_1)(x - x_2)$.

Viete's formulas give a simple relation between the roots of the quadratic polynomial and its coefficients:

$$x_1 + x_2 = \frac{-b}{a}\ , \quad x_1 \cdot x_2 = \frac{c}{a}\ .$$

- Polynomials:
  $W(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ for $x \in R$, where $n \in N$, $a_0, a_1, \ldots, a_n \in R$, $a_n \neq 0$ and a_0 - is called a constant monomial, or just a constant.

  The degree of a polynomial is the largest degree of any one term. A polynomial of degree one is called linear, of degree two is called quadratic, and of degree three is called cubic.

  The Factor Theorem

  $x - r$ is a factor of a polynomial $W(x)$ if and only if r is a root of $W(x)$.

  If $x - r$ occures k times in the factorization of $W(x)$, we say that r is a root of $W(x) = 0$ of multiplicity k.

  The integer root theorem

  If an integer is a root of a polynomial whose coefficients are integers and whose leading coefficient is ± 1, then that integer is a factor of the constant term.

  If $x_1, x_2, \ldots, x_n$ are roots of a polynomial $W(x)$ of degree n, then

$$W(x) = a_n(x - x_1)(x - x_2) \cdot \ldots \cdot (x - x_n)\ .$$

  Biquadratic equations

  A quartic equation of the form

$$ax^4 + bx^2 + c = 0$$

where $a \neq 0$ is called biquadratic equation, which is easy to solve by substitution

$$x^2 = t\ , \quad t \geq 0\ .$$

In general - equation of the form

$$ax^{2n} + bx^n + c = 0$$

where $a \neq 0$, we can solve by substitution

$$x^n = t \quad (\ t \geq 0,\ \text{when}\ n = 2k\ k \in N\ ).$$

- Rational Functions : A rational function is a quotient (ratio) of two polynomials $f(x) = \frac{W(x)}{G(x)}$, where $W(x)$ and $G(x)$ are polynomials and $G(x) \neq 0$.
  Domain: $D_f = \{x : x \in R \wedge G(x) \neq 0\}$.

  When the numerator is zero, the entire quotient is zero (provided G isn't zero at the same place): The roots of W are the roots of f. In this case, W and f have roots. When the denominator is zero, the quotient is undefined: The roots of G are the singularities of f.

$$\frac{W(x)}{G(x)} = 0 \Leftrightarrow (W(x) = 0 \wedge G(x) \neq 0)\ .$$

  Note that $f(x) = 0$ exactly if the numerator of $f(x)$ equals 0.

  Inequalities involving a rational function:

$$\frac{W(x)}{G(x)} \geq 0 \Leftrightarrow \Big(\ W(x) \cdot G(x) \geq 0 \wedge G(x) \neq 0\ \Big)\ ,$$
$$\frac{W(x)}{G(x)} \leq 0 \Leftrightarrow \Big(\ W(x) \cdot G(x) \leq 0 \wedge G(x) \neq 0\ \Big)\ .$$

  1. Find all points x where the numerator $W(x)$ equals 0. Draw a picture of the x-axis and mark these points (the critical points of the inequality).
  2. The rest of the procedure is more or less identical to the one we used for polynomial inequalities. Our critical points partition the x-axis into a few intervals. Compute sign of $W(x) \cdot G(x)$ in each interval.
  3.The roots of $G(x)$ will never be part of the set of solutions.

  REMARK

  We can only multiply an inequality by a number if we know its sign.

  But as long as you are methodical in factoring, in finding the zeros and the undefined points, and in finding the signs of each factor on each interval, you should consistently get the right answers.

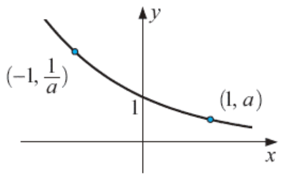
- Power function: $f(x) = x^n$

  - for $n \in N$ power function is a polynomial

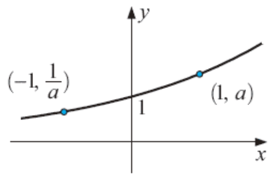
  - the domain of power function depends on n.

- Exponential function: $f(x) = a^x$, where $a \in R^+ \setminus \{1\}$, $x \in R$ is one of the most important functions in mathematics.

$0 < a < 1$



$a > 1$



$$a^{x_1} = a^{x_2} \Leftrightarrow x_1 = x_2$$

$$a \in (0,1) \Rightarrow \left(a^{x_1} > a^{x_2} \Leftrightarrow x_1 < x_2 \right)$$

$$a \in (1,\infty) \Rightarrow \left(a^{x_1} > a^{x_2} \Leftrightarrow x_1 > x_2 \right)$$

The natural exponential function is the function $y = e^x$, where e is the base of the natural logarithm also known as Euler's number. This function is written as $exp(x)$.

- Logarithmic Function: $f(x) = \log_a x$,

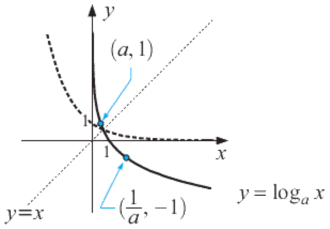
  where $a \in R^+ \setminus \{1\}$ and $x \in R^+$.

  The two most common logarithms are called common logarithms and natural logarithms:

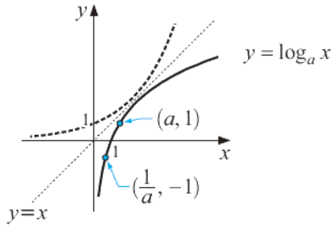
  - natural logarithms $f(x) = \ln x$ have a base of e (it means $f(x) = \log_e x$),

  - common logarithms $f(x) = \log x$ have a base of 10 (it means $f(x) = \log_{10} x$).

$0 < a < 1$



$a > 1$



$$\log_a x_1 = \log_a x_2 \Leftrightarrow x_1 = x_2$$

$$a \in (0,1) \Rightarrow \left(\log_a x_1 > \log_a x_2 \Leftrightarrow x_1 < x_2 \right)$$

$$a \in (1, \infty) \Rightarrow \left(\log_a x_1 > \log_a x_2 \Leftrightarrow x_1 > x_2 \right)$$

In mathematics, a logarithm (to base a) of a number b is the exponent c that satisfies $a^c = b$:

$$\log_a b = c \Leftrightarrow a^c = b$$

where $a \in R^+ \setminus \{1\}$, $b \in R^+$.

$$\log_a 1 = 0 \ , \quad \log_a a = 1 \ , \quad a^{\log_a b} = b$$

PROPERTIES OF LOGARITHMIC FUNCTIONS

If $a, c \in R^+ \setminus \{1\}$ and $b, b_1, b_2 \in R^+$, then

$$\log_a(b_1 \cdot b_2) = \log_a b_1 + \log_a b_2$$

$$\log_a \frac{b_1}{b_2} = \log_a b_1 - \log_a b_2$$

$$\log_a b^n = n \log_a b$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

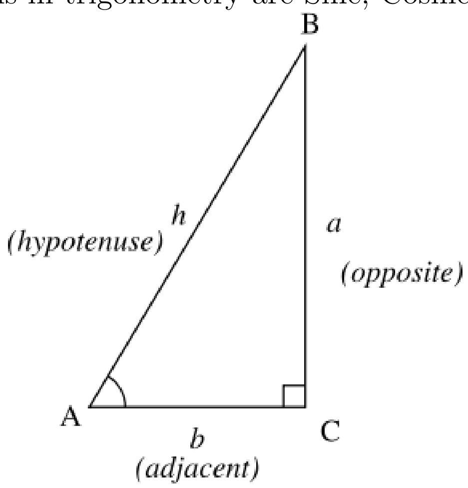
- Trigonometric functions:

In mathematics, the trigonometric functions (also called circular functions) are functions of an angle.

In navigation and astronomy we measure angles in degrees, but in calculus, it is usually best to use radians.

Trigonometric functions are commonly defined as ratios of two sides of a right triangle containing the angle.

The three main functions in trigonometry are Sine, Cosine and Tangent.



We use the following names for the sides of the triangle:

- The hypotenuse is the side opposite the right angle, or defined as the longest side of a right-angled triangle, in this case h.

- The opposite side is the side opposite to the angle we are interested in, in this case a.

- The adjacent side is the side that is in contact with the angle we are interested in and the right angle, hence its name. In this case the adjacent side is b.

- The sine function $f(x) = \sin x\ ,\quad x \in R$

  The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse.

  A sine curve is also called a sinusoid.

  $\sin(-x) = -\sin x$

  $\sin(x + 2k\pi) = \sin x,\ k \in Z$

- The cosine function $f(x) = \cos x\ ,\quad x \in R$

  The cosine of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse.

  $\cos(-x) = \cos x$

  $\cos(x + 2k\pi) = \cos x,\ k \in Z$

- The tangent function $f(x) = \tan x\ ,\quad x \in R \setminus \left\{ \frac{\pi}{2} + k\pi \right\},\quad k \in Z$

  The tangent of an angle is the ratio of the length of the opposite side to the length of the adjacent side.

  $\tan(-x) = -\tan x,\quad \tan(x + k\pi) = \tan x,\ k \in Z$

- The cotangent function $f(x) = \cot x\ ,\quad x \in R \setminus \left\{ k\pi \right\},\quad k \in Z$

  $\cot(-x) = -\cot x\ ,\quad \cot(x + k\pi) = \cot x,\ k \in Z$

Basic Trigonometric Identities

$$\tan x = \frac{\sin x}{\cos x} = \frac{1}{\cot x}\ ,\quad \tan x \cdot \cot x = 1\ ,$$

The Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1\ ,$$

Double-Angle Formulas:

$$\sin 2x = 2 \sin x \cos x\ ,\quad \cos 2x = \cos^2 x - \sin^2 x\ ,$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}\,, \quad \cos^2 x = \frac{1 + \cos 2x}{2}\,.$$

INVERSE TRIGONOMETRIC FUNCTIONS

The trigonometric functions are periodic, and hence not injective (an injective function is a function which associates distinct arguments to distinct values), so strictly they do not have an inverse function. Therefore to define an inverse function we must restrict their domains so that the trigonometric function is bijective.

For inverse trigonometric functions, the notations sin^{-1} and cos^{-1} are often used for arcsin and arccos, etc. When this notation is used, the inverse functions could be confused with the multiplicative inverses of the functions. The notation using the "arc-" prefix avoids such confusion. In computer programming languages the functions arcsin, arccos, arctan, are usually called *asin*, *acos*, *atan*.

REMARK

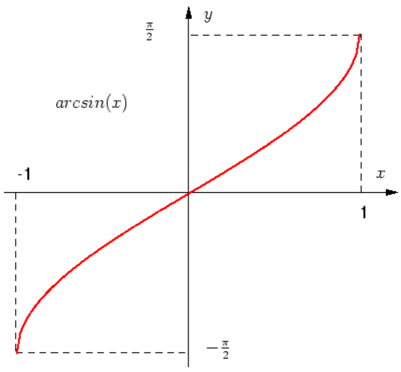
The trigonometric functions are periodic, and thus get all their values infinitely many times. Therefore their inverse functions, the cyclometric functions, are multivalued, but the values within suitable chosen intervals are unique.

REMARK

As a trigonometric function gives the length of a arc as function of an angle, the inverse cyclometric function gives the angle as function of an arc.

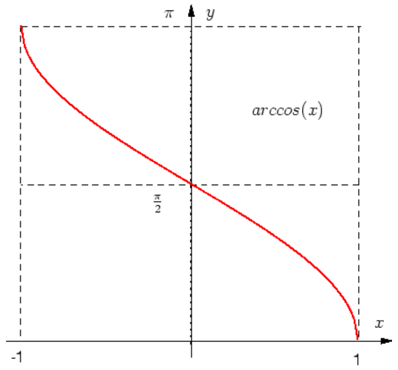
THE ARCSIN FUNCTION

We restrict the domain of the sine function to $\left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$. Now this restriction is invertible because each image value in $< -1, 1 >$ has just one origin in $\left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$. The inverse function of that restricted sine function is called the arcsine function. We write $\arcsin x$ or *asin* x. The graph $y = \arcsin x$ is the mirror image of the restricted sine graph with respect to the line $y = x$. The domain is $< -1, 1 >$ and the range is $\left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$.



THE ARCCOS FUNCTION

We restrict the domain of the cosine function to $< 0, \pi >$. Now this restriction is invertible because each image value in $< -1, 1 >$ has just one origin in $< 0, \pi >$. The inverse function of that restricted cosine function is called the arccosine function. We write $\arccos x$ or *acos* x. The graph $y = \arccos x$ is the mirror image of the restricted cosine graph with respect to the line $y = x$. The domain is $< -1, 1 >$ and the range is $< 0, \pi >$.



THE ARCTAN FUNCTION

We restrict the domain of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now this restriction is invertible because each image value in has just one origin in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The inverse function of that restricted tangent function is called the arctangent function. We write $\arctan x$ or *atan* x. The graph $y = \arctan x$ is the mirror image of the restricted tangent graph with respect to the line $y = x$. The domain is R and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

THE ARCCOT FUNCTION

We restrict the domain of the cotangent function to $(0, \pi)$. Now this restriction is invertible because each image value in has just one origin in $(0, \pi)$. The inverse function of that restricted cotangent function is called the arccotangent function. We write arccot x or acot x. The graph $y =$ arccot x is the mirror image of the restricted cotangent graph with respect to the line $y = x$. The domain is R and the range is $(0, \pi)$.