Notations - part 1

Types of Real Numbers

$N = \{\ 1, 2, 3, 4, ...\ \}$ - The set of natural numbers (unfortunately, 0 is sometimes also included in the list of "natural" numbers)

$Z = \{\ 0, \pm 1, \pm 2, \pm 3, \pm 4, ...\ \}$ - The set of integers - the integers (from the Latin *integer*, which means untouched, whole, entire) are the set of numbers including the positive natural numbers $\{1, 2, 3, \dots\}$ their negatives $\{-1, -2, -3, \dots\}$, and the number zero

$Q = \{\ \frac{p}{q}\ :\ \ p \in Z, q \in N\ \}$ - The set of rational numbers - a rational number is a number which can be expressed as a ratio of two integers

IQ - an irrational number it is a number which cannot be expressed as a fraction $\frac{m}{n}$, where m and n are integers, with n non-zero (for example $\sqrt{2}$, π)

R - The set of real numbers - the real numbers include both rational numbers and irrational numbers

List of symbols

$\in$ - "Element-of symbol" - ... is an element of a set ...

$\subset$ - "Proper subset symbol" - ... is a proper subset of ...

$\vee$ - or

$\wedge$ - and

$\cup$ - "Union symbol" - ... union ...

$\cap$ - "Intersection symbol" - ... intersected with ...

$\forall$ - "Universal quantifier" - for all ...

$\exists$ - "Existential quantifier" - there exists a(n) ...

$\Rightarrow$ - "Logical implication" - ... implies ... if ... then ...

$\Leftrightarrow$ - "Logical equivalence" - ... if and only if ...

$\approx$ - "Approximate equal sign" - ... is approximately equal to ...

(a, b) - open interval

$< a, b >$ - closed interval

$(a, b >$ - half-open interval

$\{a, b\}$ - a set

$\emptyset$ - "Null symbol" - the empty set

- The number e

  The mathematical constant e (the unique real number) is occasionally called Euler's number after the Swiss mathematician Leonhard Euler, or Napier's (Neper's) constant in honor of the Scottish mathematician John Napier who introduced logarithms.

$$e \approx 2.71828182845904523536028747135266249775724709369995957496696762772407663035354759457138217852516642742746639193200305992181 74\ldots$$

- The factorial $n!$ is the product of all positive integers less than or equal to n:

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n\ , \quad 0! = 1\ ,$$

  where $n \in N$.

  Example: $(n+1)! = \underbrace{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n}_{n!} \cdot (n+1) = n! \cdot (n+1)$.

- The binomial coefficient: $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  where $n \in N$, $k \in N$ and $k \leq n$. We have:

$$\binom{n}{0} = 1 \quad \binom{n}{n} = 1 \quad \binom{n}{1} = n$$

- Algebraic Rules for Working with Powers:

  x^n - x to the nth power

$$a^0 = 1\ , \quad a^1 = a\ , \quad a^{n+1} = a^n \cdot a$$

$$a^{-n} = \frac{1}{a^n}\ , \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$a^{n+m} = a^n \cdot a^m\ , \quad \frac{a^m}{a^n} = a^{m-n}\ , \quad (a^m)^n = a^{m \cdot n}$$

  where $a \in R \setminus \{0\}$, $m, n \in N$.

- Short Multiplication Formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

- Root Rules: if $a \geq 0$, $b \geq 0$, $m, n \in N$, then

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a}\ \sqrt[n]{b}\ , \quad \sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}\ , \quad a\ \sqrt[n]{b} = \sqrt[n]{a^n b}\ .$$

- Absolute values: Let $x, y \in R$. The absolute value or magnitude of a number x, denoted by $|x|$ (read "the absolute value of x"), is defined by formula

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

The vertical lines in the symbol $|x|$ are called absolute value bars. On a number line, $|x|$ is the distance of x is a distance of x from the orgin. It tells us how far it is from 0 to x but not the direction.

$$\sqrt{x^2} = |x|$$

Arithmetic with Absolute Values:

$$|x| \geq 0\ , \quad |x| = |-x|\ , \quad |x \cdot y| = |x| \cdot |y|\ , \quad |x + y| \leq |x| + |y|$$

If $a \geq 0$, then

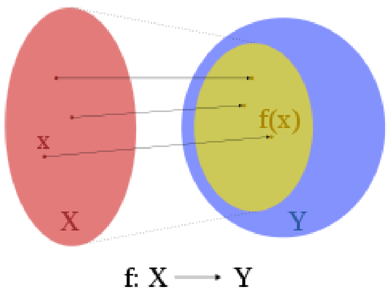
$$|x| \leq a \Leftrightarrow -a \leq x \leq a$$

$$|x| \geq a \Leftrightarrow (x \leq -a \vee x \geq a)$$

Definitions

A relation is a set of ordered pairs. The set of first entries of the ordered pairs is the domain D of the relation. The set of the second entries is the range R of the relation.

There is no universal agreement as to the definition of the range of a function. Some authors define the range of a function to be equal to the codomain, and others define the range of a function to be equal to the image.

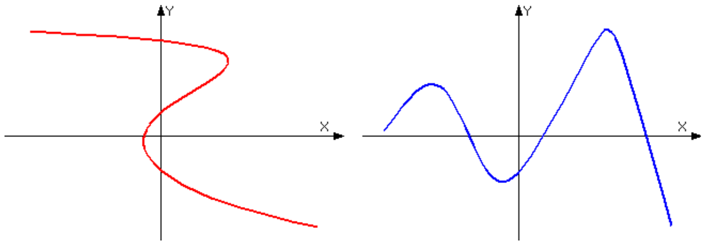


Function f is a function from domain X to codomain Y.

The smaller oval inside Y is the range of f.

Definitions

A <u>function</u> is a relation that assigns a single element of R to each element of D.



Definition

Two functions $f : D_f \to Y$ and $g : D_g \to Y$ are <u>equal</u> if and only if

$$D_f = D_g$$

and for every x from domain

$$f(x) = g(x)\ .$$

Example

Compare the domains and ranges of the functions:

a) $f(x) = \dfrac{x}{x}$ and $g(x) = \sin^2 x + \cos^2 x$,

b) $f(x) = \sqrt{x^2}$ and $g(x) = x$.

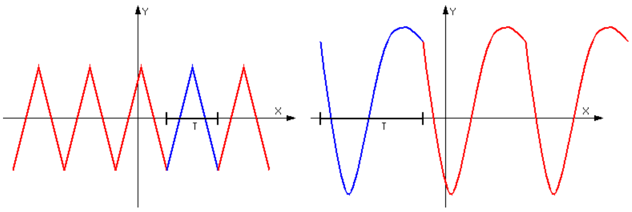
Are they equal? Explain your answer.

FUNCIONS - SOME PROPERTIES

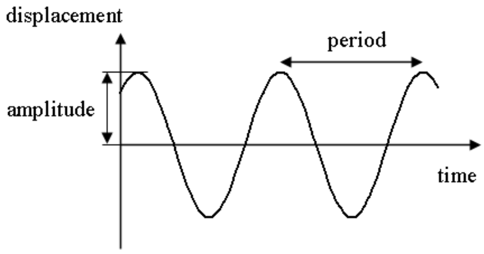
DEFINITION

A function $f : X \to Y$ is <u>periodic</u> if there is a positive number $T > 0$ such that for every value of x holds

$$x \pm T \in X \quad \textbf{and} \quad f(x+T) = f(x) \; .$$



The smallest value of T is the <u>period</u> of f.



DEFINITION

A function $f : X \to Y$ is

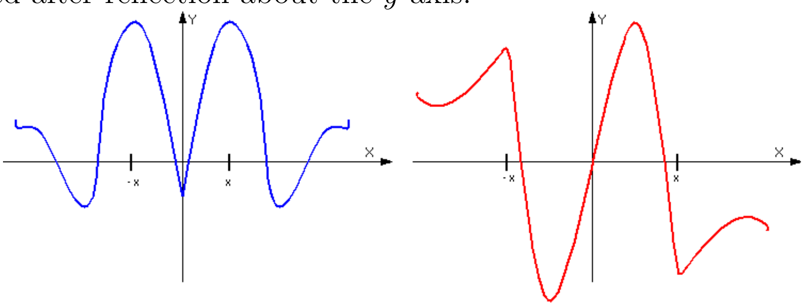
- <u>even</u> if the following equation holds for all $x \in X$:

$$-x \in X \quad \textbf{and} \quad f(-x) = f(x) \; .$$

- <u>odd</u> if the following equation holds for all $x \in X$:

$$-x \in X \quad \textbf{oraz} \quad f(-x) = -f(x) \; .$$

The only function which is both even and odd is the constant function which is identically zero. Geometrically, an even function is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the y-axis.



Geometrically, an odd function is symmetric with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

DEFINITION

A function f defined on some set X with real values is called bounded, if the set of its values is bounded. In other words, there exist numbers $m, M \in R$ such that for all $x \in X$

$$m \leq f(x) \leq M \; .$$

Sometimes, if $f(x) \leq M$ for all x in X, then the function is said to be bounded above by M.

On the other hand, if $m \leq f(x)$ for all x in X, then the function is said to be bounded below by m.

DEFINITION

A function $f : X \to Y$ defined on a subset of the real numbers with real values is called

- increasing (strictly increasing) for all $x_1, x_2 \in X$

$$(\; x_1 < x_2 \;) \Longrightarrow (\; f(x_1) < f(x_2) \;) \; .$$

- decreasing (strictly decreasing) for all $x_1, x_2 \in X$

$$(\; x_1 < x_2 \;) \Longrightarrow (\; f(x_1) > f(x_2) \;) \; .$$

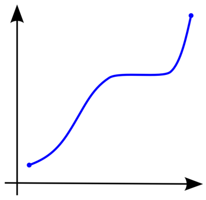
- non-decreasing for all $x_1, x_2 \in X$

$$(\; x_1 < x_2 \;) \Longrightarrow (\; f(x_1) \leq f(x_2) \;) \; .$$

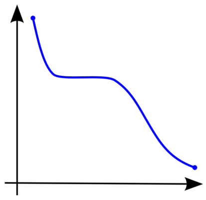
- non-increasing for all $x_1, x_2 \in X$

$$(\; x_1 < x_2 \;) \Longrightarrow (\; f(x_1) \geq f(x_2) \;) \; .$$

EXAMPLE



non-decreasing function



non-increasing function

DEFINITION

A function f is said to be surjective if its values span its whole codomain. That is, for every y in the codomain, there is at least one x in the domain such that $f(x) = y$.

Said another way, a function $f : X \to Y$ is surjective if and only if its range is equal to its codomain. A surjective function is called a surjection, and also said to be onto:

$$f : X \xrightarrow{onto} Y \ .$$

DEFINITION

A function f is said to be an injective function is a function which associates distinct arguments to distinct values. More precisely, a function $f : X \to Y$ is said to be injective if for all $x_1, x_2 \in X$

$$(\ x_1 \neq x_2\ ) \Longrightarrow (\ f(x_1) \neq f(x_2)\ )\ .$$

An injective function is called an injection, and is also said to be an information-preserving or one-to-one function. A function that is not injective is sometimes called many-to-one.

DEFINITION

A function is said to be a bijection, or a bijective function if it is both one-to-one (injective) and onto (surjective).

The inverse relation of a bijective function $f : X \to Y$ is the relation $f^{-1} : Y \to X$ defined by

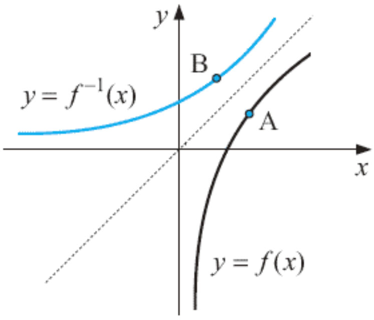
$$f^{-1}(y) = x \iff y = f(x)$$

where $x \in X,\ y \in Y$.

METHOD THAT GENERATES A FORMULA FOR THE INVERSE

1. Write $y = f(x)$
2. Solve for x in terms of y
3. Write f^{-1} for x in step 2.

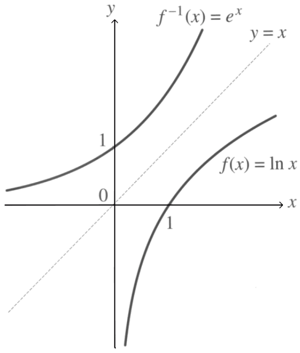
The graph of f^{-1} is obtained by simply reflecting the graph of f through the line $y = x$.



EXAMPLE

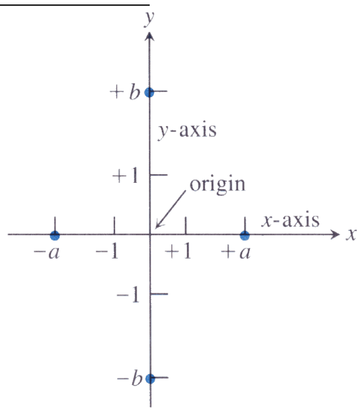
Determine whether the given function has an inverse. If an inverse exists, give the domain and range of the inverse and graph the function and its inverse.

$f(x) = \ln x$, $D : x > 0$, $W : y \in R$.

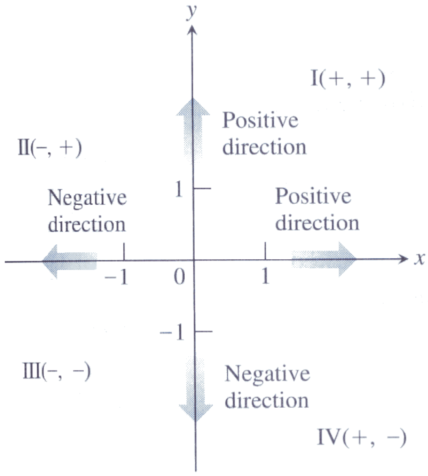


NOTATIONS - PART 2

1. Consider the point $P(a, b)$. The number a from the x-axis is the <u>x-coordinate</u> of P. The number b from the y-axis is the <u>y-coordinate</u> of P.



2. The axes divide the plane into four regions called <u>quadrants</u>.



3. Consider the points on the graph of an equation where the graph touches, or crosses, the axes. These points are the intercepts. For example - we can find the (algebraic) x intercepts by setting y equal to 0 in the equation and and solving for x.