Exercise 1. Sketch graphs of the following functions. In each case state range and domain. Write down each formula using words - not mathematical symbols.

$$\text{a) } f(x) = -x^2+1, \quad \text{b) } sgn(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}, \quad \text{c) } f(x) = \cos x, \quad \text{d) } f(x) = \tfrac{1}{x}, \quad \text{e) } f(x) = \sqrt{x-1}.$$

Exercise 2. Sketch graphs of the following functions and state images and preimages. Note the pronunciation of each new formula.

a) $f : \mathbf{R} \to \mathbf{R}, \quad f(x) = 2x+1, \quad f([0,2]) = ?, \quad f^{-1}([-1,1]) = ?$

b) $f : \mathbf{R} \to \{3\}, \quad f(x) = 3, \quad f([1,13]) = ?, \quad f^{-1}(\{3\}) = ?$

c$_1$) $f : [0,2\pi] \to [-1,1], \quad f(x) = \sin x, \quad f^{-1}([0,0.5]) = ?, \quad f([\frac{\pi}{4}, \frac{3\pi}{4}]) = ?$

c$_2$) $f : \mathbf{R} \to [-1,1], \quad f(x) = \sin x, \quad f^{-1}([0,0.5]) = ?$

d) $f : \mathbf{R}\backslash\{0\} \to \mathbf{R}\backslash\{0\}, \quad f(x) = \frac{1}{x}, \quad f^{-1}([1,2]) = ?, \quad f([-1,1]\backslash\{0\}) = ?$

e) $f : \mathbf{R} \to \{-1,0,1\}, \quad f(x) = sgn(x),$
$f^{-1}(\{1\}) = ?, \quad f^{-1}(\{-1,1\}) = ?, \quad f(\{1\}) = ?, \quad f(\{1,2,3\}) = ?, \quad f([0,2]) = ?, \quad f((0,2]) = ?$

Exercise 3. Check if the following functions are equal. Note the pronunciation of each new formula.

$$\text{a) } f(x) = \tfrac{x}{x}, \quad g(x) = \sin^2 x + \cos^2 x, \qquad \text{b) } f(x) = \sqrt{x}\sqrt{x-1}, \quad g(x) = \sqrt{x(x-1)},$$
$$\text{c) } f(x) = \tfrac{\sin x}{2} + 4, \quad g(x) = \tfrac{\sin x + 8}{2}.$$

Exercise 4. Which functions from Exercise 2 are injections, surjections and bijections and which ones are neither? Justify your opinion.

Exercise 5. List all functions $f : X \to Y$ such that

$$\text{a) } X = \{1,2,3\} \text{ and } Y = \{a,b\}, \qquad \text{b) } X = \{a,b\} \text{ and } Y = \{1,2,3\}.$$

In each case state if the function is an injection, a surjection, a bijection or neither. Why are there no injections in example (a)? Why are there no surjections in example (b)? If we dealt with sets $[1,3]$ and $[a,b]$ – would we be able to find injections and surjections then?

Exercise 6. Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ such that it is

a) an injection, but not a surjection, b) a surjection, but not an injection,
c) neither an injection nor a surjection, d) both an injection and a surjection.

Exercise 7. Check if $f : \mathbf{Z} \to \mathbf{Z}$ is a surjection or an injection if

$$\text{a) } f(x) = 2x-3, \qquad \text{b) } f(x) = x^2 - 5x, \qquad \text{c) } f(x) = |x| - 1.$$

How would the answers change if $f : \mathbf{N} \to \mathbf{N}$? Would all of these functions be well-defined?

Exercise 8. Find the inverses of the following functions and sketch their graphs.

$$\text{a) } y = 2x+1, \qquad \text{b) } y = \tfrac{x}{3} - 8, \qquad \text{c) } y = \tfrac{1}{x}, \qquad \text{d) } y = \sqrt{x}, \qquad \text{e) } y = \sqrt{x-1}.$$

Exercise 9. Give an example of a function that

a) is even, b) is odd, c) is neither even nor odd, d) is both even and odd.

Exercise 10. Odd or even? Check the following functions using the definition.

a) $f(x) = \sin x$, b) $f(x) = \cos x$, c) $f(x) = sgn(x)$,

d) $f(x) = x^5 + 2x^2$, e) $f(x) = \frac{1}{|x|}$, f) $f(x) = (x+2)^2$.

Exercise 11. Show an example (a graph and a formula) of a function that is

a) non-increasing, b) non-decreasing, c) increasing, d) decreasing.

Exercise 12. Draw a graph of a function that has the following properties

a) domain: $(-\infty, +\infty)\setminus\{0\}$, range: $[-1, 2]$, is decreasing, is neither even nor odd,

b) domain and range: $\mathbf{R}$, increasing, not a surjection, odd,

c) domain: $\mathbf{R}$, range: $(2, \infty)$, not strictly monotonic, even.

Exercise 13. Sketch graphs of the following functions. Write down exact formulas in examples (a) and (b).

a) $f(x) = x^2$,
$f(x-1)$, $f(x+1)$, $f(x-1)+2$, $f(x-1)-2$, $|f(x-1)-2|$, $f(|x-1|)-2$, $|-f(-|x-1|)-2|$,

b) $f(x) = \sin x$,
$f(x-\pi)$, $-f(x+\pi)$, $f(x+\pi)-1$, $|f(x+\pi)|-1$, $|f(-|x|)-1|+1$, $3f(4x)$, $-3f(-0.5x)$,

c) $f(x) = -||||x|-1|-1|-1|$.

Exercise 14. Find $f(g(x))$, $g(f(x))$ and $g(g(x))$. In each case state domain and range.

a) $f(x) = x+1,\ g(x) = x-1$,

b) $f(x) = \sqrt{x},\ g(x) = x^2 - 2$,

c) $f(x) = \sin x,\ g(x) = x + \pi$,

d) $f(x) = \frac{2}{x},\ g(x) = \frac{3}{x}$,

e*) $f(x) = x + \pi,\ g(x) = \sin x$.