DEFINITION (*The Cauchy "epsilon-delta" definition*)

The limit of $f(x)$ as x approaches c is the number L:

$$\lim_{x\to c} f(x) = L$$

if for given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about c such that for all x

$$(0 < |x - c| < \delta) \Longrightarrow (|f(x) - L| < \varepsilon)$$

REMARK

The above definition actually says, that the closer x gets to c, the closer $f(x)$ gets to L.

REMARK

Sometimes the values of a function $f(x)$ tend to different limits as x approaches number c from different sides. When this happens, we call the limit of f as x approaches c from the right the right-hand limit of f at c:

$$\lim_{x\to c^+} f(x)$$

and the limit of f as x approaches c form the left the left-hand limit of f at c:

$$\lim_{x\to c^-} f(x)$$

We sometimes call $\lim_{x\to c} f(x)$ the two-sided limit of f at c to distinguish it from the one-sided right-hand and left-hand limits of f at c.

RELATION BETWEEN ONE-SIDED AND TWO-SIDED LIMITS

A function $f(x)$ has a limit at point c if and only if the right-hand and left-hand limits at c exist and are equal.

In symbols:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad (\lim_{x\to c^+} f(x) = L \quad \wedge \quad \lim_{x\to c^-} f(x) = L)$$

EXAMPLES

Example 1. $\lim_{x\to 0}(\frac{x^3-2x+1}{x-1} + 1) = \frac{0^3-2\cdot 0+1}{0-2} + 1 = -\frac{1}{2} + 1 = \frac{1}{2}$.

Example 2. $\lim_{x\to 9} \frac{6-2\sqrt{x}}{x-9}$ =?. Here we cannot just substitute 9 for x, because the domain of the function does not include number 9. Furthermore, by putting 9 instead of x we would have ended up with $\frac{0}{0}$. Instead, we'll transform the formula of the function using the short multiplication formula $a^2 - b^2 = (a-b)(a+b)$:

$$\lim_{x\to 9}\frac{6-2\sqrt{x}}{x-9}=\left[\frac{0}{0}\right]=\lim_{x\to 9}\frac{-2(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x}+3)}=\lim_{x\to 9}\frac{-2}{\sqrt{x}+3}=-\frac{1}{3}.$$

Example 3. Another similar example:

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x^3-1}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x^3-1)(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{x+3-4}{(x^3-1)(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{x-1}{(x^3-1)(\sqrt{x+3}-2)}=\dots$$

We will now use the fact that $x^3-1=(x-1)(x^2+x+1)$:

$$\dots=\lim_{x\to 1}\frac{x-1}{(x-1)(x^2+x+1)(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{1}{(x^2+x+1)(\sqrt{x+3}+2)}=\frac{1}{12}.$$

LIMITS IN INFINITY – THE PROPER LIMIT (*The Cauchy "epsilon-delta" definition*)

The limit of $f(x)$ as x approaches ∞ is the number L (different from $\pm\infty$):

$$\lim_{x\to\infty} f(x)=L$$

if for given any radius $\varepsilon>0$ about L there exists a number $a\in\mathbf{R}$ such that

$$(x>a)\Longrightarrow(|f(x)-L|<\varepsilon)$$

The limit of $f(x)$ as x approaches $-\infty$ is the number L (different from $\pm\infty$):

$$\lim_{x\to-\infty} f(x)=L$$

if for given any radius $\varepsilon>0$ about L there exists a number $a\in\mathbf{R}$ such that

$$(x<a)\Longrightarrow(|f(x)-L|<\varepsilon)$$

LIMITS – THE IMPROPER LIMIT AT A POINT (*The Cauchy "epsilon-delta" definition*)

The limit of $f(x)$ as x approaches c is ∞:

$$\lim_{x\to c} f(x)=\infty$$

if for given any number $M>0$ there exists a radius $\delta>0$ about c such that for all x

$$(0<|x-c|<\delta)\Longrightarrow(f(x)>M)$$

The limit of $f(x)$ as x approaches c is $-\infty$:

$$\lim_{x\to c} f(x)=-\infty$$

if for given any number $m<0$ there exists a radius $\delta>0$ about c such that for all x

$$(0<|x-c|<\delta)\Longrightarrow(f(x)<m)$$

LIMITS THAT APPEAR OFTEN

$$\lim_{x\to 0}\tfrac{\sin x}{x}=1 \qquad \lim_{x\to\pm\infty}(1+\tfrac{1}{x})^x=e \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

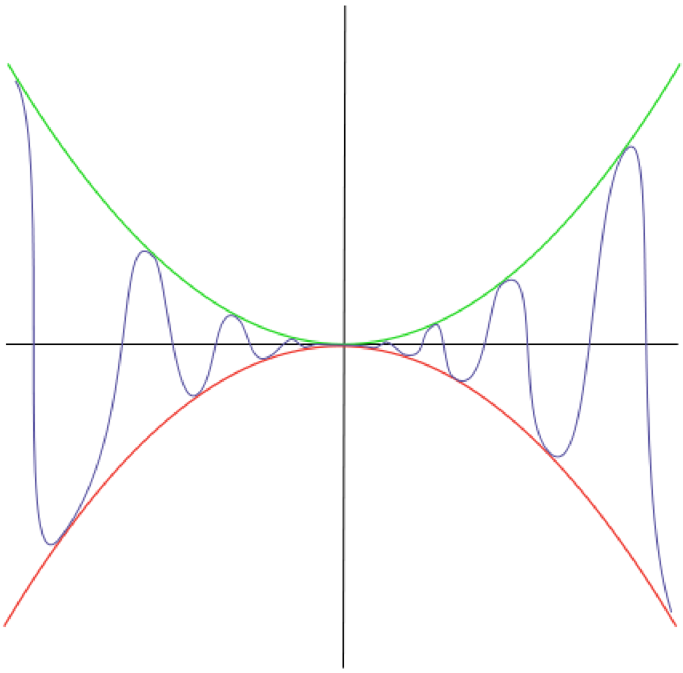
THE SANDWICH THEOREM

Suppose that

$$f(x)\le g(x)\le h(x)$$

for all $x\neq c$ in some interval about c and that $f(x)$ and $h(x)$ approach the same limit L as x approaches c. Then

$$\lim_{x\to c} g(x)=L$$



THE LIMIT COMBINATION THEOREM

If $\lim\limits_{x\to c} f(x)=L_1$ and $\lim\limits_{x\to c} g(x)=L_2$, where $L_1, L_2\in\mathbf{R}$, then

- $\lim\limits_{x\to c}(f(x)\pm g(x))=\lim\limits_{x\to c} f(x)\pm\lim\limits_{x\to c} g(x)$,
- $\lim\limits_{x\to c}(a\cdot f(x))=a\cdot(\lim\limits_{x\to c} f(x))$, where $a\in R$,
- $\lim\limits_{x\to c}(f(x)\cdot g(x))=(\lim\limits_{x\to c} f(x))\cdot(\lim\limits_{x\to c} g(x))$,
- $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$, if $\lim\limits_{x\to c} g(x)\neq 0$.

The Combination Theorem for Limits at infinity is analogous to the corresponding theorem for limits when $x \to c$.

REMARK

If function f has a proper limit 0 in infinity and function g is bounded, then the function $f(x) \cdot g(x)$ also has limit 0 in infinity (*see Example 9*).

MORE EXAMPLES

Example 4. $\lim\limits_{x\to 0} \frac{2x}{\tan 3x} = \lim\limits_{x\to 0} \frac{2x \cdot \cos 3x}{\sin 3x} = \lim\limits_{x\to 0} \frac{2\cos 3x}{\frac{\sin 3x}{x}} = \lim\limits_{x\to 0} \frac{2\cos 3x}{3 \cdot \frac{\sin 3x}{3x}} = \frac{2}{3}$.

Example 5. $\lim\limits_{x\to 0} \frac{\sin 7x + \sin 3x}{\sin 5x - 4x} = \lim\limits_{x\to 0} \frac{x(\frac{\sin 7x}{x} + \frac{\sin 3x}{x})}{x(\frac{\sin 5x}{x} - 4)} = \lim\limits_{x\to 0} \frac{7 \cdot \frac{\sin 7x}{7x} + 3 \cdot \frac{\sin 3x}{3x}}{5 \cdot \frac{\sin 5x}{5x} - 4} = \frac{7+3}{5-4} = 10$.

Example 6. $\lim\limits_{x\to -\infty} (-4x^3 + 5x^2 - \frac{6}{x^2} + \frac{7}{x^3}) = \lim\limits_{x\to -\infty} x^3(-4 + \frac{5}{x} - \frac{6}{x^2} + \frac{7}{x^6}) = \infty$.

Example 7. $\lim\limits_{x\to -\infty} \frac{5x^3 - x^2 + 1}{1 - 3x^2} = \lim\limits_{x\to -\infty} \frac{x^3(5 - \frac{1}{x} + \frac{1}{x^3})}{x^2(\frac{1}{x^2} - 3)} = \lim\limits_{x\to -\infty} \frac{x(5 - \frac{1}{x} + \frac{1}{x^3})}{\frac{1}{x^2} - 3} = [\frac{-\infty}{-3}] = \infty$.

Example 8. $\lim\limits_{x\to -\infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 - 4x}) = [\infty - \infty] = \lim\limits_{x\to -\infty} \frac{(\sqrt{x^2+2x} - \sqrt{x^2-4x})(\sqrt{x^2+2x} + \sqrt{x^2-4x})}{\sqrt{x^2+2x} + \sqrt{x^2-4x}} =$ $\lim\limits_{x\to -\infty} \frac{6x}{\sqrt{x^2+2x} + \sqrt{x^2-4x}} = \lim\limits_{x\to -\infty} \frac{6x}{|x|(\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{4}{x}})} = \lim\limits_{x\to -\infty} \frac{6x}{-x(\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{4}{x}})} = \frac{6}{-(1+1)} = -3$.

Example 9. $\lim\limits_{x\to\infty} \frac{x^2}{2x^3 - 7} \cdot \cos 3x =?$ We notice that the function is a product of two functions: $f(x) = \frac{x^2}{2x^3-7}$ and $g(x) = \cos 3x$. Function $g(x)$ is bounded from below by -1 and from above by +1. Furthermore, $\lim\limits_{x\to\infty} f(x) = 0$. Hence, $\lim\limits_{x\to\infty} \frac{x^2}{2x^3-7} \cdot \cos 3x = 0$.

Example 10. $\lim\limits_{x\to\infty} \frac{4^{x+2} + 5\cdot 3^x + 2^x}{-5\cdot 2^{2x} + 3^{x-2} + 1} =?$ First we need to simplify the formula of the function, so that the exponent of each number is exactly x:

$$\lim_{x\to\infty} \frac{4^{x+2} + 5\cdot 3^x + 2^x}{-5\cdot 2^{2x} + 3^{x-2} + 1} = \lim_{x\to\infty} \frac{16\cdot 4^x + 5\cdot 3^x + 2^x}{-5\cdot 4^x + \frac{1}{9}\cdot 3^x + 1} = \dots$$

and the next step is to divide both the numerator and the denominator by 4^x (because it is the highest number raised to the power of x):

$$\dots = \lim_{x\to\infty} \frac{16 + 5\cdot(\frac{3}{4})^x + (\frac{1}{2})^x}{-5 + \frac{1}{9}\cdot(\frac{3}{4})^x + (\frac{1}{4})^x} = -\frac{16}{5}$$

We also used the fact that $\lim\limits_{x\to\infty} q^x = 0$ for $|q| < 1$.

Example 11. $\lim\limits_{x\to\infty} (\frac{2x+1}{2x-5})^{4x} = \lim\limits_{x\to\infty} (\frac{2x(1+\frac{1}{2x})}{2x(1-\frac{5}{2x})})^{4x} = \lim\limits_{x\to\infty} \frac{(1+\frac{1}{2x})^{4x}}{(1-\frac{1}{\frac{2x}{5}})^{4x}} = \lim\limits_{x\to\infty} \frac{[(1+\frac{1}{2x})^{2x}]^2}{[(1-\frac{1}{\frac{2x}{5}})^{-\frac{2x}{5}}]^{-10}} = \frac{e^2}{e^{-10}} =$ e^{12}.

Example 12. $\lim\limits_{x\to\infty} (\frac{3x-4}{3x+2})^{\frac{x+1}{3}} = \lim\limits_{x\to\infty} (\frac{3x+2-6}{3x+2})^{\frac{x+1}{3}} = \lim\limits_{x\to\infty} (1 - \frac{6}{3x+2})^{\frac{x+1}{3}} = \lim\limits_{x\to\infty} [(1 - \frac{1}{\frac{3x+2}{6}})^{-\frac{3x+2}{6}}]^{\frac{-6}{3x+2}\cdot\frac{x+1}{3}} =$ $\lim\limits_{x\to\infty} [(1 - \frac{1}{\frac{3x+2}{6}})^{-\frac{3x+2}{6}}]^{-\frac{2x+2}{3x+2}} = \lim\limits_{x\to\infty} e^{-\frac{2x+2}{3x+2}} = e^{-\frac{2}{3}}$.

Example 13. $\lim\limits_{x\to -\infty} \cos(\arctan x) = \cos(\lim\limits_{x\to -\infty} \arctan x) = \cos\frac{-\pi}{2} = 0$.

Example 14. $\lim\limits_{x\to 0} \ln\left(\frac{\sin ex}{x}\right) = \ln\left(\lim\limits_{x\to 0} \frac{\sin ex}{x}\right) = \ln\left(\lim\limits_{x\to 0} e \cdot \frac{\sin ex}{ex}\right) = \ln(e\cdot 1) = \ln e = 1.$

ONE-SIDED LIMITS – EXAMPLES

Example 15a. $\lim\limits_{x\to 3^+} \frac{x-7}{9-x^2} = \left[\frac{-4}{0^-}\right] = \infty.$

Example 15b. $\lim\limits_{x\to 3^-} \frac{x-7}{9-x^2} = \left[\frac{-4}{0^+}\right] = -\infty.$

Example 16a. $\lim\limits_{x\to 0+} e^{\frac{1}{x}} = [x \to 0^+ \implies \frac{1}{0^+} \to \infty \implies e^{\infty} \to \infty] = \infty$.

Example 16b. $\lim\limits_{x\to 0-} e^{\frac{1}{x}} = [x \to 0^- \implies \frac{1}{0^-} \to -\infty \implies e^{-\infty} \to 0] = 0$.

Example 17a. $\lim\limits_{x\to 0^+} \frac{1}{1-\pi^{\frac{1}{x}}} = [x \to 0^+ \implies \frac{1}{0^+} \to \infty \implies \pi^{\infty} \to \infty] = [\frac{1}{1-\infty}] = 0$.

Example 17b. $\lim\limits_{x\to 0^-} \frac{1}{1-\pi^{\frac{1}{x}}} = [x \to 0^- \implies \frac{1}{0^-} \to -\infty \implies \pi^{-\infty} \to 0] = [\frac{1}{1-0}] = 1$.

Example 18. Check if $\lim\limits_{x\to 2} \frac{\sin(2x-4)}{|x-2|}$ exists. If so, what is its value?

Solution: Let us consider two cases:

(a) $\lim\limits_{x\to 2^-} \frac{\sin(2x-4)}{|x-2|} = \lim\limits_{x\to 2^-} \frac{\sin(2x-4)}{-(x-2)} = \lim\limits_{x\to 2^-} -2 \cdot \frac{\sin(2(x-2))}{2(x-2)} = -2\cdot 1 = -2,$

(b) $\lim\limits_{x\to 2^+} \frac{\sin(2x-4)}{|x-2|} = \lim\limits_{x\to 2^+} \frac{\sin(2x-4)}{(x-2)} = \lim\limits_{x\to 2^+} 2 \cdot \frac{\sin(2(x-2))}{2(x-2)} = 2\cdot 1 = 2.$

Since $\lim\limits_{x\to 2^-} \frac{\sin(2x-4)}{|x-2|} \neq \lim\limits_{x\to 2^+} \frac{\sin(2x-4)}{|x-2|}$, then the limit does not exist.