Definition – Function Continuity

A function $y = f(x)$ is continuous at an interior point c of its domain if $\lim_{x\to c} f(x) = f(c)$.

A function $y = f(x)$ is continuous at a left endpoint a of its domain if $\lim_{x\to a^+} f(x) = f(a)$.

A function $y = f(x)$ is continuous at a right endpoint b of its domain if $\lim_{x\to b^-} f(x) = f(b)$.

A function is continuous if it is continuous at each point of its domain.

Definition

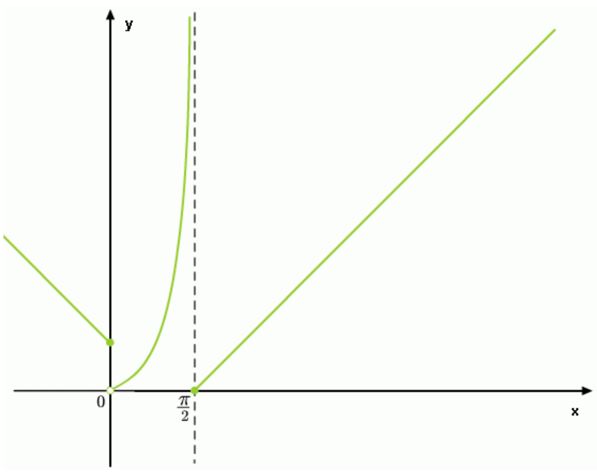
If a function $y = f(x)$ is not continuous at a point c, we say that f is discontinuous at c and call c a point of discontinuity of f.

Remark

A function may happen to be continuous in only one direction, either from the "left" or from the "right". A right-continuous function is a function which is continuous at all points when approached from the right. Likewise a left-continuous function is a function which is continuous at all points when approached from the left:

$$\lim_{x\to c^+} f(x) = f(c), \quad \lim_{x\to c^-} f(x) = f(c)$$

A function is continuous if and only if it is both right-continuous and left-continuous.



REMARK

If a function has a domain which is not an interval, the notion of a continuous function as one whose graph you can draw without taking your pencil off the paper is not quite correct.

Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{\sin x}{x}$. Neither function is defined at $x = 0$, so each has domain $\mathbf{R} \setminus \{\mathbf{0}\}$ of real numbers except 0, and each function is continuous. The question of continuity at $x = 0$ does not arise, since it is not in the domain.

Algebraic combinations of continuous functions are continuous at every point at which they are defined.

TYPES OF DISCONTINUITY

There are two types of discontinuity:

1. If

$$\lim_{x \to c^-} f(x) \neq f(c) \quad \textbf{or} \quad \lim_{x \to c^+} f(x) \neq f(c)$$

and both limits exist, then:

1(a) if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(x_0)$ then the graph of $f(x)$ has a hole at $x = c$ (discontinuity is removable),

1(b) if limits are unequal, then the point c is called a jump-point and the function has a jump or a saltus at that point: $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$.

2. If $\lim_{x \to c^-} f(x) = \pm\infty$ **or** $\lim_{x \to c^+} f(x) = \pm\infty$ then the graph of $f(x)$ has an infinite discontinuity (then $f(x)$ has a vertical asymptote at $x = c$).

An essential discontinuity is one which isn't of three previous types - at least one of the one-sided limits doesn't exist.

REMARK

We can sometimes extend the domain of a function f to include more points where it is continuous. If c is a point where f is not defined but where $\lim_{x \to c} f(x)$ exists, we can define $f(c)$ to be the value of the limit. The extended f is automatically continuous becouse $f(c)$ exists and equals $\lim_{x \to c} f(x)$. This function is called the continuous extension of the original function to the point $x = c$.

FACTS ABOUT CONTINUOUS FUNCTIONS

1. If two functions f and g are continuous, then

- $f + g$
- $f - g$
- $a \cdot f$, where $a \in \mathbf{R}$
- $f \cdot g$
- if $g(x) \neq 0$ for all x in the domain, then $\frac{f}{g}$

are also continuous.

2. The composition $f \circ g$ of two continuous functions is continuous.

THE INTERMEDIATE VALUE THEOREM

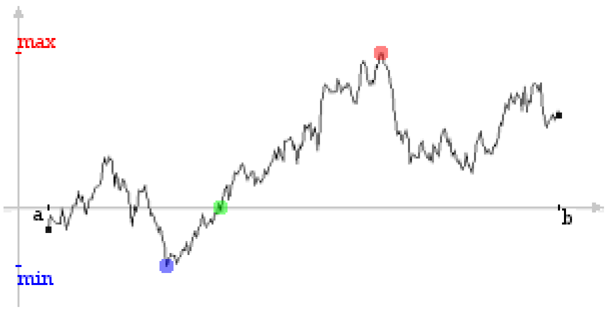
The intermediate value theorem is an existence theorem based on the real number property of completeness, and states:

If the real-valued function f is continuous on the closed interval $[a, b]$ and k is some number between $f(a)$ and $f(b)$, then there is at least one point c between a and b such that $f(c) = k$.

For example, if a child grows from 1m to 1.5m between the ages of 2 years and 6 years, then, at some time between 2 years and 6 years of age, the child's height must have been 1.25m.

A continuous function on a closed interval has a maximum and a minimum, and assumes all values between them.

Example 1. If $f(a) \cdot f(b) < 0$, then there must be at least one point c in (a, b) where $f(c) = 0$.



These statements are false if the function is defined on an open interval (a, b) (or any set that is not both closed and bounded), as for example the continuous function $f(x) = \frac{1}{x}$ defined on the open interval $(0, 1)$.

Examples

Example 2. Check the continuity of the function $f(x) = \frac{x+1}{|x-2|}$.

Solution: The domain of our function is $\mathbf{R}\backslash\{2\}$, furthermore:

$$f(x) = \begin{cases} -\frac{x+1}{x-2} & x < 2 \\ \frac{x+1}{x-2} & x > 2 \end{cases}$$

Therefore, the function is continuous in each of intervals $(-\infty, 2)$ and $(2, \infty)$, but it is not continuous in $x_0 = 2$ because it is not defined in that point. So, function $f(x)$ is continuous in its domain.

Example 3. Check the continuity of the following function and determine types of discontinuity (if they exist):

$$f(x) = \begin{cases} 5 & x < -2 \\ (\frac{1}{2})^x + 1 & -2 \leq x \leq 0 \\ \log_{\frac{1}{2}}(x + \frac{1}{2}) & 0 < x \leq \frac{3}{2} \\ \frac{-2}{2x-3} & x > \frac{3}{2} \end{cases}$$

Solution: Firstly, we need to check the domain of two "pieces" of function $f(x)$:

- $x + \frac{1}{2} > 0$ in $(0, \frac{3}{2}]$,
- $2x - 3 \neq 0$ in $(\frac{3}{2}, \infty)$.

Both conditions are satisfied in given intervals, so function $f(x)$ is continuous in

$$(-\infty, -2) \cup (-2, 0) \cup (0, \frac{3}{2}) \cup (\frac{3}{2}, \infty).$$

Therefore, our points of interest are: $x_0 = -2, x_1 = 0, x_2 = \frac{3}{2}$.

- We notice that $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} 5 = 5$ and $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} ((\frac{1}{2})^x + 1) = 5$, so in general $\lim\limits_{x\to -2} f(x) = 5$. Moreover, $f(-2) = 5$. To sum up: $\lim\limits_{x\to -2} = f(-2) = 5$ and $f(x)$ is continuous in $x_0 = -2$.

- Let us now check $x_1 = 0$: we can easily see that $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} ((\frac{1}{2})^x + 1) = 2$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \log_{\frac{1}{2}}(x + \frac{1}{2}) = 1$. The two limits do exist but they are not equal! The limit $\lim\limits_{x\to 0} f(x)$ does not exist and function $f(x)$ has a discontinuity of the first type in $x_1 = 0$.

- The last point is $x_2 = \frac{3}{2}$:

$$\lim_{x\to \frac{3}{2}^-} f(x) = \lim_{x\to \frac{3}{2}^-} \log_{\frac{1}{2}}(x + \tfrac{1}{2}) = \log_{\frac{1}{2}}(\tfrac{3}{2} + \tfrac{1}{2}) = \log_{\frac{1}{2}} 2 = -1 = f(\tfrac{3}{2}),$$
$$\lim_{x\to \frac{3}{2}^+} f(x) = \lim_{x\to \frac{3}{2}^+} \tfrac{-2}{2x-3} = -\infty \neq -1.$$

  The left-side limit is proper, while the right-side limit is not – therefore $x_2 = \frac{3}{2}$ is a point of discontinuity of the second type.

Example 4. Determine the value of parameter α for which function

$$f(x) = \begin{cases} (x-\alpha)^2 & x < 1 \\ 2^x - 1 & x \geq 1 \end{cases}$$

is continuous for all $x \in \mathbf{R}$.

Solution: Function $f(x)$ is continuous everywhere in $(-\infty, 1) \cup (1, \infty)$, so our only point of interest is $x_0 = 1$. We know that $f(1) = 2^1 - 1 = 1$, so let us establish the values of limits:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-\alpha)^2 = (1-\alpha)^2,$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2^x - 1) = 1.$$

The function will be continuous in $x_0 = 1$ if and only if $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$. Therefore, we need to solve the following equation: $(1-\alpha)^2 = 1$, to which answers are: $\alpha = 0 \vee \alpha = 2$. Function $f(x)$ will be continuous in $\mathbf{R}$ for $\alpha \in \{0, 2\}$.