Definite integral

Let $f(x)$ be continuous and integrable in $[a,b]$. A definite integral of function $f(x)$ in the interval of $[a,b]$ is given by the following formula:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is the antiderivative of $f(x)$ (in other words: the primitive of $f(x)$).

From now on, integral $\int f(x)dx$ should be considered as the indefinite integral.

Properties

$$\int_a^a f(x)dx = 0, \qquad \int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Examples

Example 1. $\int_2^4 x^2dx = \frac{x^3}{3}\Big|_2^4 = \frac{4^3}{3} - \frac{2^3}{3} = \frac{64-8}{3} = \frac{56}{3}.$

Example 2. Integration by substitution.

$$\int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{2}dx = \left|\begin{array}{c} t = \frac{x}{2} \\ dt = \left(\frac{x}{2}\right)' = \frac{dx}{2} \\ 2dt = dx \\ t(\frac{\pi}{2}) = \frac{\pi}{4},\ t(\pi) = \frac{\pi}{2} \ \ !!! \end{array}\right| = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos t dt = 2 \sin t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2(1 - \tfrac{\sqrt{2}}{2}) = 2 - \sqrt{2}.$$

Example 3. Integration by parts.

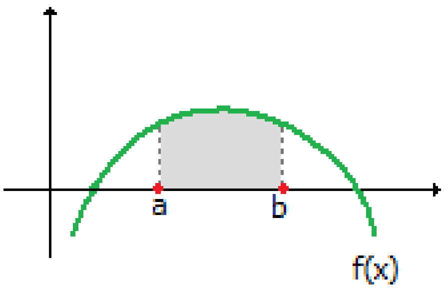
$$\int_1^e x\ln x dx = \left|\begin{array}{cc} f(x) = \ln x & f'(x) = \frac{1}{x} \\ g'(x) = x & g(x) = \frac{x^2}{2} \end{array}\right| = \left(\frac{x^2\ln x}{2}\right)\Big|_1^e - \int_1^e \frac{x^2}{2x}dx = \left(\frac{e^2}{2} - 0\right) - \int_1^e \frac{x}{2}dx =$$

$$= \frac{e^2}{2} - \frac{x^2}{4}\Big|_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = \frac{e^2+1}{4}.$$

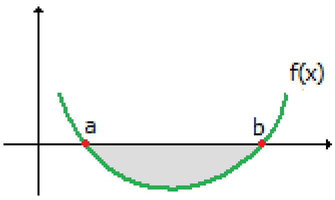
Applications - Area

If $f(x) \geq 0$ for $x \in [a,b]$ then $\int_a^b f(x)dx$ is equal to the area bounded by the graph of $f(x)$, lines $x = a$ and $x = b$ and the OX axis. Here are some useful templates for calculating areas:

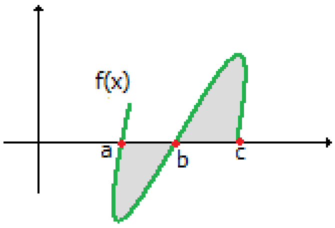
$\int_a^b f(x)dx$

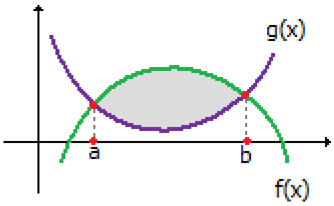


$\int_a^b -f(x)dx$

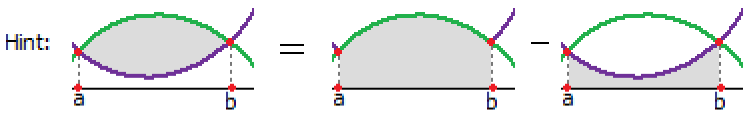


$\int_a^b -f(x)dx + \int_b^c f(x)dx$

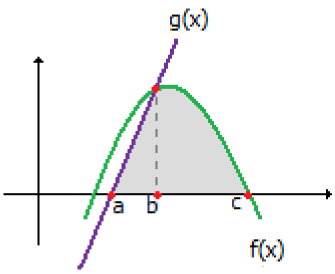




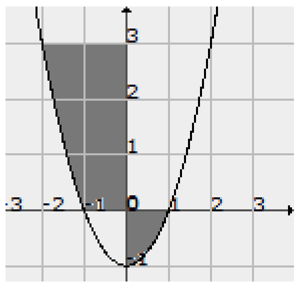
$\int_a^b f(x)dx - \int_a^b g(x)dx$



$\int_a^b g(x)dx + \int_b^c f(x)$



Example 1. Calculate the shaded area of the $f(x) = x^2 - 1$ graph.

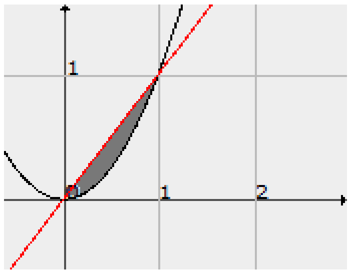


Solution: The area is equal to

$(\int_{-2}^{0} 3dx - \int_{-2}^{-1}(x^2-1)dx) - \int_0^1 (x^2-1)dx =$

$= 3x|_{-2}^{0} - (\frac{x^3}{3} - x)|_{-2}^{-1} - (\frac{x^3}{3} - x)|_0^1 =$

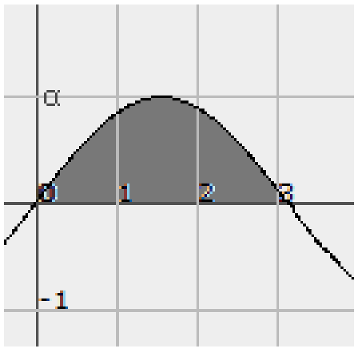
$= (0+6) - (-\frac{1}{3} + 1 - (-\frac{8}{3} + 2)) - (\frac{1}{3} - 1) = 5\frac{1}{3}.$



Example 2. Calculate the area bounded by graphs of $g(x) = x^2$ and $f(x) = x$. Make a drawing.

Solution: The area is equal to

$\int_0^1 (x - x^2)dx = (\frac{x^2}{2} - \frac{x^3}{3})|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.



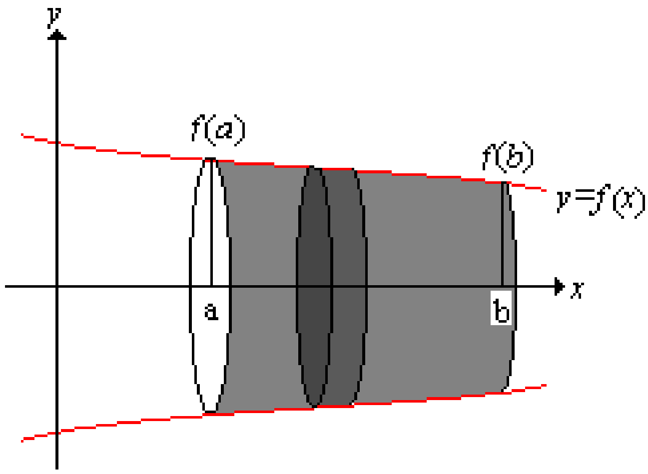
Example 3. Let R be a region bounded by the graphs of $\alpha \sin x$ and $y = 0$ for $x \in \{0, \pi\}$. For which values of parameter α is the area of R equal to 4? Sketch a diagram.

Solution: The area is equal to

$\int_0^\pi \alpha \sin x dx = -\alpha \cos x|_0^\pi = -\alpha(-1 - 1) = 2\alpha = 4$.

Therefore $\alpha = 2$.

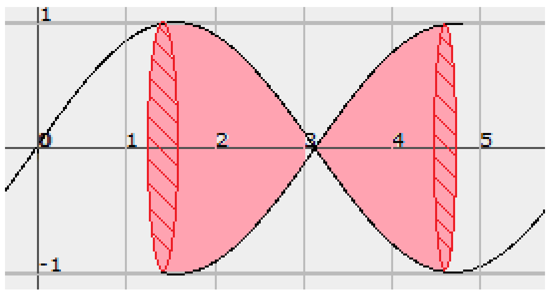
APPLICATIONS - VOLUME



If a space figure (in other words: a solid) is created by revolvong a graph of some function about the OX axis, then it is called a solid of revolution.

The volume of such a solid is given by the following formula:

$$V = \pi \int_a^b (f(x))^2 dx.$$



Example 4. Draw a solid of a revolution of function $f(x) = \sin x$ in the interval of $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Calculate the volume of the solid.

Solution: $V = \pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2 x dx = \pi(\frac{\sin 2x + 2x}{4})_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \pi(\frac{0+3\pi}{4} - \frac{0+\pi}{4}) = \frac{\pi^2}{2}$.

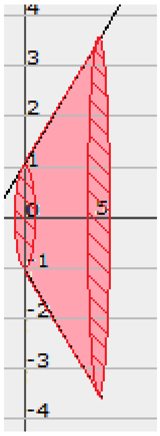
APPLICATIONS - SURFACE OF REVOLUTION

The surface of a solid of revolution obtained by rotating the graph of $f(x)$ about the OX axis in an interval of $[a, b]$ is given by the following formula:

$$S = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}dx.$$

Note that this formula describes only the side-surface. To obtain the complete surface, one needs to add $\pi f^2(a) + \pi f^2(b)$ to S.

Example 5. Draw a solid of a revolution of a function $f(x) = \frac{x}{2} + 1$ in the interval of $[0, 5]$ and then calculate its surface. What is the complete surface of this solid?



Solution: The side-surface is equal to:
$S = 2\pi \int_0^5 (\frac{x}{2} + 1)\sqrt{1 + (\frac{1}{2})^2}dx = 2\pi \int_0^5 \frac{\sqrt{5}}{2}(\frac{x}{2} + 1)dx = \sqrt{5}\pi(\frac{x^2}{4} + x)|_0^5 = \sqrt{5}\pi(\frac{25}{4} + 5 - 0) = \frac{56\sqrt{5}\pi}{5}$.
Therefore, the complete surface is equal to
$S_c = \pi(\frac{56\sqrt{5}}{5} + 1 + (3\frac{1}{2})^2) = \pi(\frac{56\sqrt{5}}{5} + \frac{53}{4})$.

APPLICATIONS - LENGTH OF THE CURVE

Let C be a curve of $f(x)$ in the interval of $[a, b]$. Lhe length of C is given by the formula:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$