The Idea

We already know that graphs of some functions rise very *steeply* while graphs of other functions rise more *gently*. Differentiation is a method to compute the rate at which a quantity y changes with respect to the change in another quantity x upon which it is dependent. This rate of change is called the derivative of y with respect to x.

Equations for lines

The point-slope equation of the line through the point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$.
The slope-intercept equation of the line with slope m and y-intercept b is $y = mx + b$.

The Derivative of a Function

The derivative of a function f is the function f' whose value at x_0 is defined by the equation

$$f'(x_0) \stackrel{def}{=} \lim_{\Delta x \to o} \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

whenever the limit exists (**note**: sometimes it is worthwhile to check both right-side and left-side limits!).

Differentiable at a Point

A function that has a derivative at a point x is said to be differentiable at x.

Differentiable Function

A function that is differentiable at every point of its domain is said to be differentiable.

Remark

The branch of mathematics that deals with derivatives is called differential calculus.

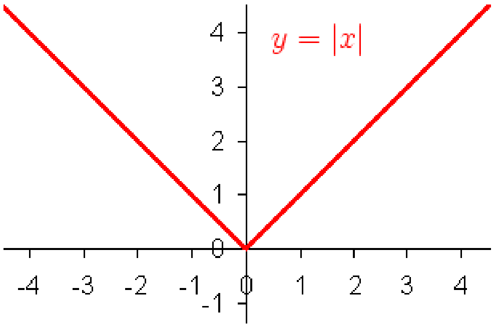
Examples

Example 1. Calculate the derivative of $f(x) = |x|$ at $x_0 = 0$ using the definition.

Solution: We need to calculate both righ-side and left-side limits at $x_0 = 0$:

$$f'_-(0) = \lim_{\Delta x \to 0^-} \frac{|\Delta x| - 0}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x \to 0^-} (-1) = -1,$$
$$f'_+(0) = \lim_{\Delta x \to 0^+} \frac{|\Delta x| - 0}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0^+} (1) = 1.$$

We can see that $f'_-(0) \neq f'_+(0)$, so $f(x)$ is not differentiable at $x_0 = 0$. It is a very interesting example, because $f(x)$ is continuous at that point! Actually, $f(x) = |x|$ is not differentiable at $x_0 = 0$ because its graph has a **sharp corner** at that point.



Example 2. Calculate the derivative of $f(x) = x^3$ for any $x \in \mathbf{R}$.

Solution: We will use property $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. Using the definition, we obtain:

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x - x)((x+\Delta x)^2 + (x+\Delta x)x + x^2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + (\Delta x)^2 + 3x\Delta x)}{\Delta x} =$$
$$\lim_{\Delta x \to 0} (3x^2 + (\Delta x)^2 + 3x\Delta x) = 3x^2.$$

Example 3. Calculate the derivative of $f(x) = \sin x$ for $x_0 = \frac{\pi}{2}$.

Solution: This time we will use trigonometric identity $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$:

$$f'\left(\tfrac{\pi}{2}\right) = \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\pi}{2}+\Delta x\right) - \sin\frac{\pi}{2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sin\frac{\Delta x}{2}\cos\frac{\pi+\Delta x}{2}}{\Delta x} = \lim_{\Delta x \to x} \left(\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\frac{\pi+\Delta x}{2}\right) =$$
$$\lim_{\Delta x \to 0} \left(1 \cdot \cos\frac{\pi+\Delta x}{2}\right) = \cos\frac{\pi+0}{2} = \cos\frac{\pi}{2} = 0.$$

Continuity and differentiability

If $y = f(x)$ is differentiable at c, then f must also be continuous at c.

If a function is continuous at a point, it need not be differentiable there.

Remark

We have already seen in Example 1 that a function with a sharp corner at x_0 is not differentiable at x_0. What is important, is that a function with a "smooth" graph need not be differentiable! For instance the function $y = \sqrt[3]{x}$ is not differentiable at $x = 0$, because at that point its tangent is vertical (which is forbidden).

DIFFERENTIATION RULES

Derivative of a Constant: $(a)' = 0, \quad a \in \mathbf{R}.$

Power Rule: $(x^n)' = nx^{n-1}, \quad x > 0,\ n \in \mathbf{R}.$

Derivatives of elementary functions:

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x,$$
$$(\tan x)' = \tfrac{1}{\cos^2 x}, \quad x \neq \tfrac{\pi}{2} + k\pi,$$
$$(\cot x)' = -\tfrac{1}{\sin^2 x}, \quad x \neq k\pi,$$
$$(a^x)' = a^x \ln a, \quad (e^x)' = e^x,$$
$$(\log_a x)' = \tfrac{1}{x \ln a}, \quad 0 < a \neq 1,\ x > 0,$$
$$(\ln x)' = \tfrac{1}{x}, \quad x > 0,$$
$$(\arcsin x)' = \tfrac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1),$$
$$(\arccos x)' = -\tfrac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1),$$
$$(\arctan x)' = \tfrac{1}{1+x^2}, \quad (\text{arccot } x)' = -\tfrac{1}{1+x^2}$$

THEOREM

If f and g are differentiable at a point x_0, then:

- The Sum and Difference Rule:

$$(f+g)'(x_0) = f'(x_0) + g'(x_0), \qquad (f-g)'(x_0) = f'(x_0) - g'(x_0).$$

- The Constant Multiple Rule:

$$(cf)'(x_0) = cf'(x_0), \text{ where } c \in R.$$

- The Product Rule:

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

- The Quotient Rule:

$$\left(\frac{f}{g}\right)' = \tfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}, \text{ if } g(x_0) \neq 0.$$

EXAMPLES

Example 4. $(\sqrt[5]{x}-\sqrt[3]{x}+\sqrt{x}-\sqrt{2})'=(x^{\frac{1}{5}}-x^{\frac{1}{3}}+x^{\frac{1}{2}}-\sqrt{2})'=\frac{1}{5}x^{-\frac{4}{5}}-\frac{1}{3}x^{-\frac{2}{3}}+\frac{1}{2}x^{-\frac{1}{2}}+0=\frac{1}{5\sqrt[5]{x^4}}-\frac{2}{3\sqrt[3]{x^2}}+\frac{1}{2\sqrt{x}}$.

Example 5. $(-5x^2+x-\frac{3}{x}+13)'=-5(x^2)'+(x)'+(\frac{3}{x})'+(13)'=-10x+1+\frac{3}{x^2}+0=-10x+1+\frac{3}{x^2}$.

Example 6. $(2^x-\ln x+\log x)'=2^x\ln 2-\frac{1}{x}+\frac{1}{x\ln 10}$.

Example 7. $(3e^x\cos x)'=(3e^x)'\cdot\cos x+3e^x\cdot(\cos x)'=3e^x\cos x-3e^x\sin x=3e^x(\cos x-\sin x)$.

Example 8. $(\frac{x}{1+x^2})'=\frac{x'\cdot(1+x^2)-(1+x^2)'\cdot x}{(1+x^2)^2}=\frac{1\cdot(1+x^2)-(0+2x)\cdot x}{(1+x^2)^2}=\frac{-x^2+1}{(1+x^2)^2}$.

THEOREM - THE CHAIN RULE

Suppose that $f\circ g$ is the composite of the differentiable functions $y=f(u)$ and $u=g(x)$. Then $f\circ g$ is a differentiable function of x whose derivative at each value of x is

$$\Big(f(g(x))\Big)'=f'\Big(g(x)\Big)g'(x).$$

The Chain Rule is probably the most widely used differentation rule in mathematics.

EXAMPLES

Example 9. $(\sin 3x)'=?$. Function $\sin 3x$ is a composite of functions $g(x)=3x$ and $f(x)=\sin x$. Therefore: $(\sin 3x)'=\cos 3x\cdot(3x)'=\cos 3x\cdot 3=3\cos 3x$.

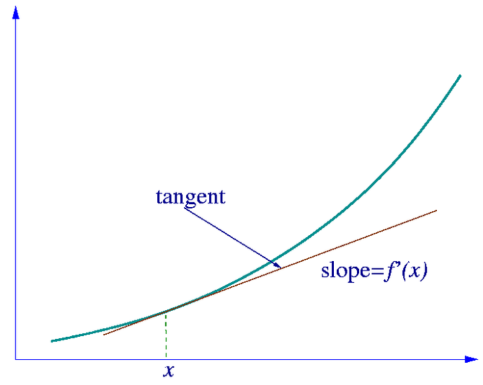
Example 10. $((2x+3)^4)'=?$. Function $(2x+3)^4)$ is a composite of functions $g(x)=(2x+3)$ and $f(x)=x^4$. Therefore: $((2x+3)^4)'=4(2x+3)^3\cdot(2x+3)'=4(2x+3)^3\cdot 2=8(2x+3)^3$.

Example 11. $(\ln(x^4+1))'=?$. Function $\ln(x^4+1)$ is a composite of functions $g(x)=x^4+1$ and $f(x)=\ln x$. Therefore: $(\ln(x^4+1))'=\frac{1}{x^4+1}\cdot(x^4+1)'=\frac{4x^3}{x^4+1}$.

Example 12. $(\arctan(\ln(x^4+1)))'=?$. Function $\arctan(\ln(x^4+1))$ is a composite of three functions $g(x)=x^4+1$, $f(x)=\ln x$ and $h(x)=\arctan x$ — we deal with $(h(f(g(x))))'$. Therefore: $(\arctan(\ln(x^4+1)))'=\frac{1}{1+(\ln(x^4+1))^2}\cdot(\ln(x^4+1))'=\frac{\frac{4x^3}{x^4+1}}{1+(\ln(x^4+1))^2}=\frac{4x^3}{(1+(\ln(x^4+1))^2)(x^4+1)}$.

TANGENT LINES

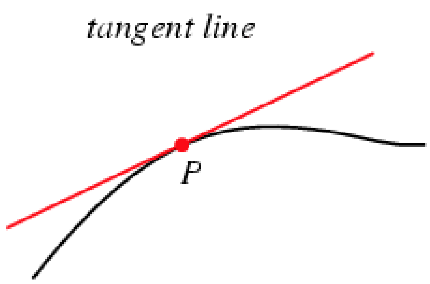
If x and y are real numbers, and if the graph of y is plotted against x, the derivative measures the slope of this graph at each point.



In plane geometry, a line is tangent to a curve, at some point, if both line and curve pass through the point with the same direction. Such a line is called the tangent line (or tangent). The tangent line is the best straight-line approximation to the curve at that point.

DEFINITION

A straight line is tangent to a given curve $y = f(x)$ at a point x_0 on the curve if the line passes through the point $P(x_0, f(x_0))$ on the curve and has slope $f'(x_0)$, where $f'(x)$ is the derivative of $f(x)$. This line is called a tangent line, or sometimes simply a tangent.



The equation of a tangent to curve $y = f(x)$ at point $P = (x_0, f(x_0))$ is:

$$y - f(x_0) = f'(x_0)(x - x_0),$$

and the equation of a normal line (i.e. a line perpendicular to the tangent line) to curve $y = f(x)$ at point $P = (x_0, f(x_0))$ is:

$$y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0),$$

Example 13. Formulate the equation of a tangent and a normal to curve $y = x^3$ at $x_0 = 2$.

Solution: $f(2) = 8$, so point $P = (2, 8)$. Also, $f'(x) = 3x^2$, so $f'(2) = 12$. The equation of a tangent line at point $(2, 8)$ is:

$$y - 8 = 12(x - 2),$$
$$y = 12x - 24 + 8,$$
$$y = 12x - 16.$$

The equation of a normal line at point $(2, 8)$ is:

$$y - 8 = \frac{-1}{12}(x - 2),$$
$$y = \frac{-x}{12} + \frac{1}{6} + 8,$$
$$y = \frac{-x}{12} + 8\frac{1}{6}.$$

Higher order derivatives

The process of differentiation can be applied several times in succession, leading in particular to the second derivative $f^{(2)}$ (or f'') of the function f, which is just the derivative of the derivative f'. The nth derivative of $f(x)$ is denoted by $f^{(n)}(x)$.

Definition

$$f^{(n)}(c) \stackrel{def}{=} [f^{(n-1)}]'(c) \quad \text{for} \quad n > 1 \text{ , where } \quad f^{(1)}(c) \stackrel{def}{=} f'(c) \quad \text{and} \quad f^{(0)}(c) \stackrel{def}{=} f(c).$$

Remark

The second derivative often has a useful physical interpretation. For example, if $f(t)$ is the position of an object at time t, then $f'(t)$ is its speed at time t and $f^{(2)}$ is its acceleration at time t.

We will need to be careful with the "non-prime" notation for derivatives. Consider each of the following:

$$f^{(2)} = f''(x),$$
$$f^2(x) = \big(f(x)\big)^2.$$

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Collectively the second, third, fourth, etc. derivatives are called higher order derivatives.

Example 14. $(x^2 + 2x + 3)'' = (2x + 2)' = 2.$

Example 15. $(\ln x)'' = (\frac{1}{x})' = \frac{-1}{x^2}.$

Total differential

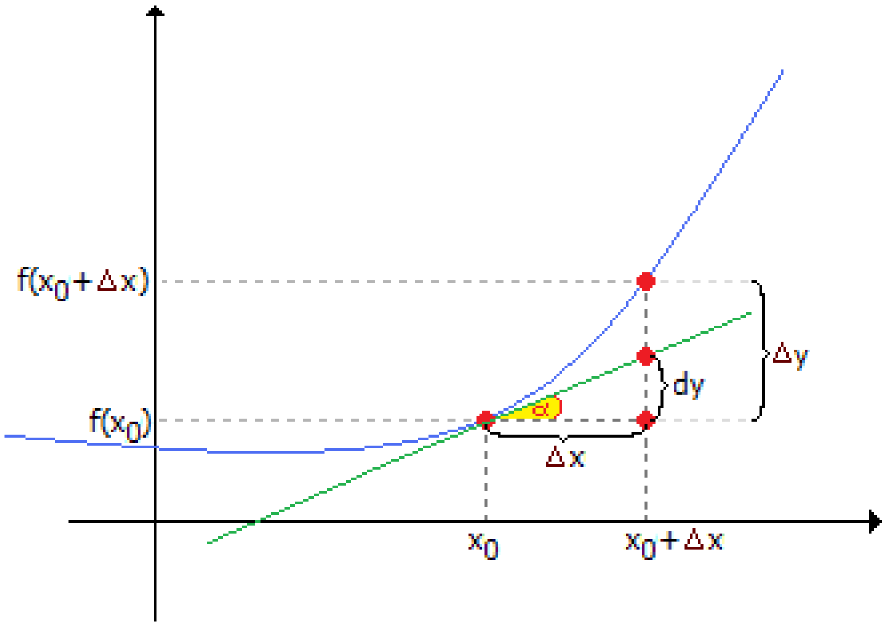
The following statements are true:

$\tan\alpha = \frac{dy}{\Delta x} = f'(x_0)$,

$f'(x_0) = f(x_0) + \Delta y$,

$f'(x_0) \approx f(x_0) + dy$,

$f'(x_0) \approx f(x_0) + f'(x_0)\cdot\Delta x$.



Definition

An expression $f'(x_0)\cdot\Delta x$ is called the total differential of function $f(x)$ at point x_0 with Δx increase. The total differential is commonly used to calculate approximated values and for calculating errors.

Example 16. Knowing that function $f(x) = x^2$ takes on value $f(25) = 625$, calculate its approximated value at point $x_0 + \Delta x = 25.02$.

Solution: We know that $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x$ and $f'(x) = 2x$. Therefore

$$f(25.05) \approx f(25) + f'(25)\cdot 0.02 = 625 + 50\cdot 0.02 = 625 + 1 = 626.$$

The exact value of $(25.02)^2$ is 626.0004, therefore the error is $|626 - 626.0004| = 000.4$.

Example 17. Using the data from the previous example check that the larger the Δx, the larger the error – calculate $(25.04)^2$ and $(25.06)^2$ using the same technique and compare errors.

Solution: a) $f(25.04) \approx 625 + 50\cdot 0.04 = 625 + 2 = 627$ and the exact value is $f(25.04) = 627.0016$. Therefore, the error is equal to 0.0016.

b) $f(25.06) \approx 625 + 50\cdot 0.06 = 625 + 3 = 628$ and the exact value is $f(25.06) = 628.0036$. Therefore, the error is equal to 0.0036.

Example 18. Calculate the approximated value of $\sqrt[3]{0.98}$ using the total differential. Compare your result with the exact value and calculate the error.

Solution: Let us take $f(x) = \sqrt[3]{x}$. Then, $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$. We notice that $x_0 = 1.0$ and $\Delta x = -0.02$. Therefore:

$$f(0.98) \approx f(1.0) + f'(1.0) \cdot \Delta x = 1 + \tfrac{1}{3} \cdot (-0.02) = 1 - \tfrac{2}{300} = 0.993(3).$$

The exact value of $\sqrt[3]{0.98}$ (up to 10 decimal digits) is 0.9932883883. Therefore, the error is equal to $|0.993(3) - 0.9932883883| = 0.000044945$.

l'Hospital's Rules

l'Hôpital's rule (also spelled l'Hospital) uses derivatives to help compute limits with indeterminate forms. Application (or repeated application) of the rule often converts an indeterminate form to a determinate form, allowing easy computation of the limit. The rule is named after the 17th-century French mathematician Guillaume de l'Hôpital, who published the rule in his book. The rule is believed to be the work of Johann Bernoulli since l'Hôpital, a nobleman, paid Bernoulli a retainer of 300F (F-French franc) per year to keep him updated on developments in calculus and to solve problems he had.

Theorem

If f and g are differentiable in a neighborhood of $x = c$, and

$$\lim_{x\to c} f(c) = \lim_{x\to c} g(c) = 0 \quad \text{or} \quad \lim_{x\to c} f(c) = \lim_{x\to c} g(c) = \pm\infty,$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \ ,$$

provided the limit on the right exists.

The same result holds for one-sided limits.

If f and g are differentiable and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \pm\infty,$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}\ ,$$

provided the last limit exists.

Example 19. $\lim\limits_{x\to0^+}\frac{e^x-e^{-x}}{\ln(\cos x)}=[\frac{0}{0}]\overset{H}{=}\lim\limits_{x\to0^+}\frac{(e^x-e^{-x})'}{(\ln(\cos x))'}=\lim\limits_{x\to0^+}\frac{e^x+e^{-x}}{-\tan x}=-\infty.$

Example 20. $\lim\limits_{x\to1^+}\frac{\tan\frac{\pi x}{2}}{\log(x-1)}=[\frac{\infty}{\infty}]\overset{H}{=}\lim\limits_{x\to1^+}\frac{(\tan\frac{\pi x}{2})'}{(\log(x-1))'}=\lim\limits_{x\to1^+}\frac{\frac{\pi}{2\cos^2\frac{\pi x}{2}}}{\frac{1}{(x-1)\ln10}}=\lim\limits_{x\to1^+}\frac{\pi(x-1)\ln10}{2\cos^2\frac{\pi x}{2}}=[\frac{\infty}{\infty}]=$ $\lim\limits_{x\to1^+}\frac{(\pi(x-1)\ln10)'}{(2\cos^2\frac{\pi x}{2})'}=\lim\limits_{x\to1^+}\frac{\pi\ln10}{-2\sin(\pi x)\cdot\frac{\pi}{2}}=-\ln10\cdot\lim\limits_{x\to1^+}\frac{1}{\sin(\pi x)}=[\frac{1}{0^-}]=-\infty.$

Remark

Many other indeterminate forms can be calculated using l'Hôpital's rule.

Indeterminate form	Rules for changing	Indeterminate form
$0\cdot\infty$	$f\cdot g=\dfrac{f}{\frac{1}{g}}$	$\frac{0}{0}$; $\frac{\infty}{\infty}$
$\infty-\infty$	$f-g=\dfrac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{fg}}$	$\frac{0}{0}$
1^∞; ∞^0; 0^0	$f^g=e^{\ln f^g}=e^{g\ln f}$	$0\cdot\infty$

Example 21. $\lim\limits_{x\to-\infty}x\cdot(\text{arccot}x-\pi)=[\infty\cdot0]=\lim\limits_{x\to-\infty}\frac{\text{arccot }x-\pi}{\frac{1}{x}}=[\frac{0}{0}]\overset{H}{=}\lim\limits_{x\to-\infty}\frac{(\text{arccot }x-\pi)'}{(\frac{1}{x})'}=$ $\lim\limits_{x\to-\infty}\frac{\frac{-1}{1+x^2}}{-\frac{1}{x^2}}=\lim\limits_{x\to-\infty}\frac{x^2}{x^2+1}=1.$

Example 22. $\lim\limits_{x\to\infty}((x+3)\cdot e^{\frac{1}{x}}-x)=[\infty-\infty]=\lim\limits_{x\to\infty}[x\cdot((1+\frac{3}{x})\cdot e^{\frac{1}{x}}-1)]=[\infty\cdot0]=$ $\lim\limits_{x\to\infty}\frac{(1+\frac{3}{x})\cdot e^{\frac{1}{x}}-1}{\frac{1}{x}}=[\frac{0}{0}]\overset{H}{=}\lim\limits_{x\to\infty}\frac{((1+\frac{3}{x})\cdot e^{\frac{1}{x}}-1)'}{(\frac{1}{x})'}=\lim\limits_{x\to\infty}\frac{-\frac{3}{x^2}\cdot e^{\frac{1}{x}}+(1+\frac{3}{x})\cdot e^{\frac{1}{x}}\cdot(-\frac{1}{x^2})}{-\frac{1}{x^2}}=\lim\limits_{x\to\infty}[e^{\frac{1}{x}}(4+\frac{3}{x})]=4.$