ELEMENTARY MATHEMATICS

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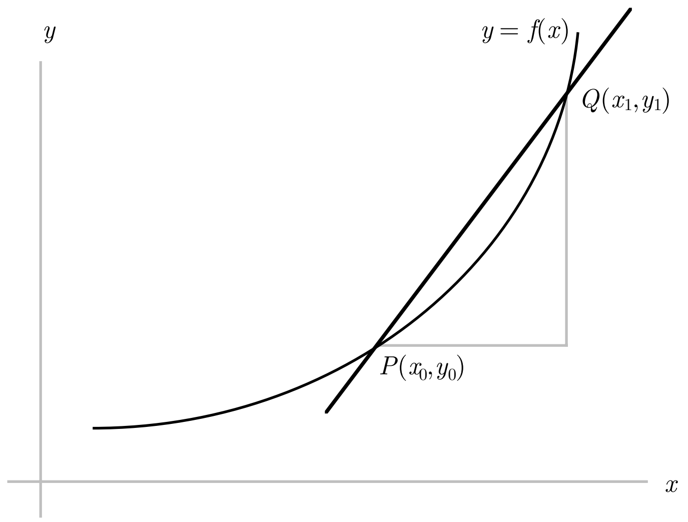
Chapter 11

INTRODUCTION TO DIFFERENTIATION

11.1. Tangent to a Curve

Consider the graph of a function $y = f(x)$. Suppose that $P(x_0, y_0)$ is a point on the curve $y = f(x)$. Consider now another point $Q(x_1, y_1)$ on the curve close to the point $P(x_0, y_0)$. We draw the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$, and obtain the picture below.

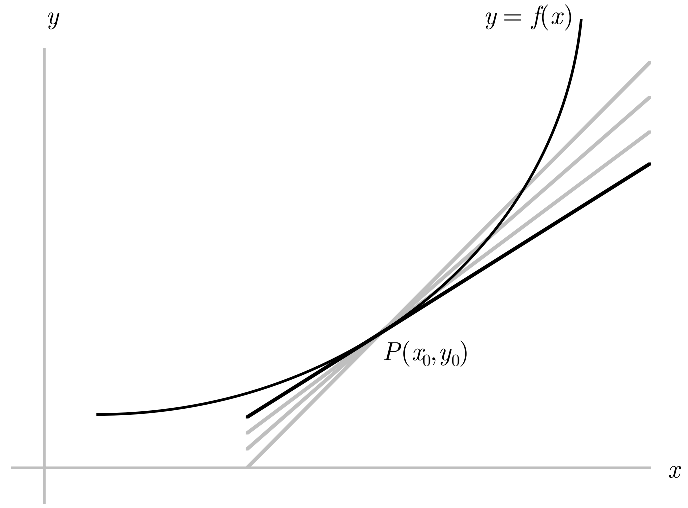


Clearly the slope of this line is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

† This chapter was written at Macquarie University in 1999.

Now let us keep the point $P(x_0, y_0)$ fixed, and move the point $Q(x_1, y_1)$ along the curve towards the point P. Eventually the line PQ becomes the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$, as shown in the picture below.

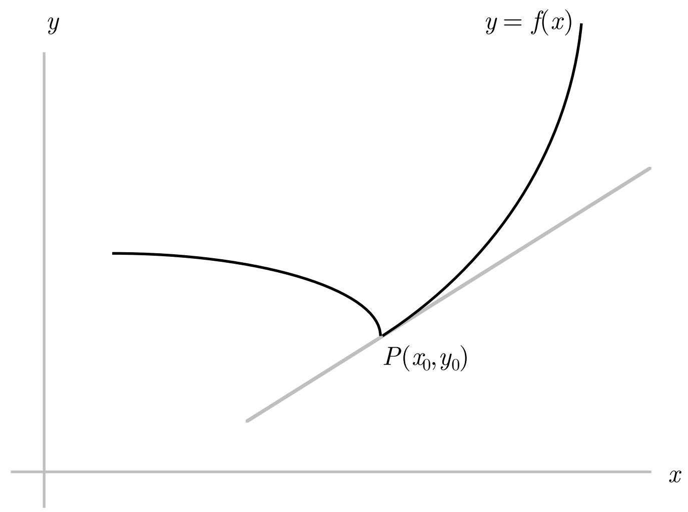


We are interested in the slope of this tangent line. Its value is called the derivative of the function $y = f(x)$ at the point $x = x_0$, and denoted by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} \qquad \text{or} \qquad f'(x_0).$$

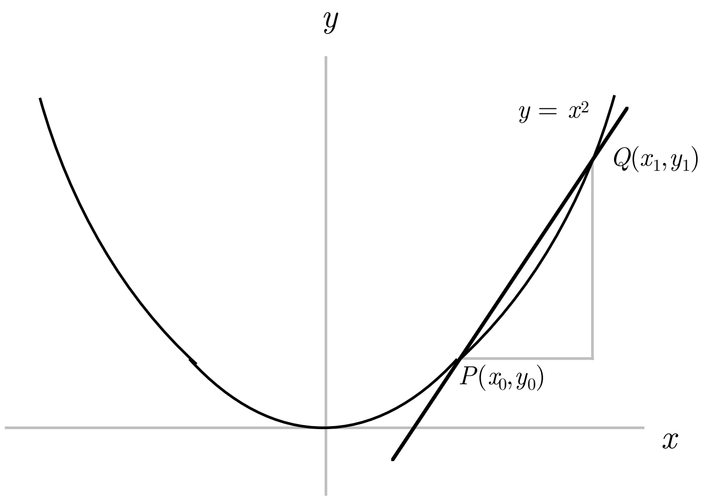
In this case, we say that the function $y = f(x)$ is differentiable at the point $x = x_0$.

Remark. Sometimes, when we move the point $Q(x_1, y_1)$ along the curve $y = f(x)$ towards the point $P(x_0, y_0)$, the line PQ does not become the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$. In this case, we say that the function $y = f(x)$ is not differentiable at the point $x = x_0$. An example of such a situation is given in the picture below.



Note that the curve $y = f(x)$ makes an abrupt turn at the point $P(x_0, y_0)$.

EXAMPLE 11.1.1. Consider the graph of the function $y = f(x) = x^2$.



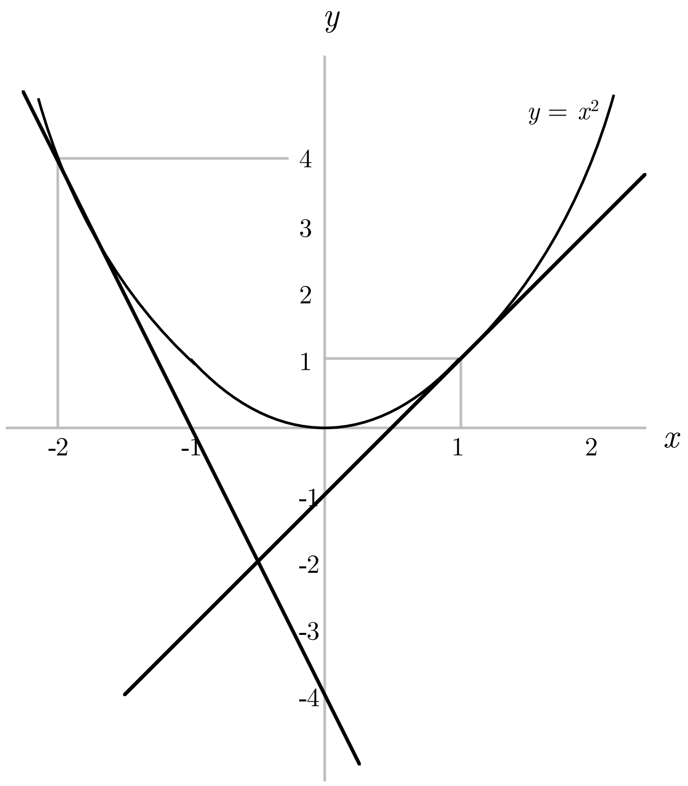
Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = x_1 + x_0.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to $x_0 + x_0 = 2x_0$. Hence for the function $y = f(x) = x^2$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 2x_0.$$

In particular, the tangent to the curve at the point $(1, 1)$ has slope 2 and so has equation $y = 2x - 1$, whereas the tangent to the curve at the point $(-2, 4)$ has slope -4 and so has equation $y = -4x - 4$.



EXAMPLE 11.1.2. Consider the graph of the function $y = f(x) = x^3$. Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^3 - x_0^3}{x_1 - x_0} = x_1^2 + x_1x_0 + x_0^2.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to $x_0^2 + x_0x_0 + x_0^2 = 3x_0^2$. Hence for the function $y = f(x) = x^3$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 3x_0^2.$$

In particular, the tangent to the curve at the point $(0, 0)$ has slope 0 and so has equation $y = 0$, whereas the tangent to the curve at the point $(2, 8)$ has slope 12 and so has equation $y = 12x - 16$.

EXAMPLE 11.1.3. Consider the graph of the function $y = f(x) = x$. Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will remain equal to 1. Hence for the function $y = f(x) = x$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 1.$$

EXAMPLE 11.1.4. Consider the graph of the function $y = f(x) = x^{1/2}$, defined for all real numbers $x \geq 0$. Suppose that $x_0 > 0$ and $x_1 > 0$. Then the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^{1/2} - x_0^{1/2}}{x_1 - x_0} = \frac{1}{x_1^{1/2} + x_0^{1/2}}.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to

$$\frac{1}{x_0^{1/2} + x_0^{1/2}} = \frac{1}{2x_0^{1/2}} = \frac{1}{2}x_0^{-1/2}.$$

Hence for the function $y = f(x) = x^{1/2}$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = \frac{1}{2}x_0^{-1/2}.$$

The above four examples are special cases of the following result.

DERIVATIVES OF POWERS. *Suppose that n is a fixed non-zero real number. Then for the function $y = f(x) = x^n$, we have*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nx^{n-1}$$

for every real number x for which x^{n-1} is defined.

Here and henceforth, we shall slightly abuse our notation and refer to $f'(x)$ as the derivative of the function $y = f(x)$, and write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

EXAMPLE 11.1.5. For the function $y = f(x) = x^{1/4}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \frac{1}{4}x^{-3/4}$$

for every positive real number x.

The rule concerning derivatives of powers does not apply in the case $n = 0$.

DERIVATIVES OF CONSTANTS. *Suppose that $f(x) = c$, where c is a fixed real number. Then $f'(x) = 0$ for every real number x.*

11.2. Arithmetic of Derivatives

Very often, we need to find the derivatives of complicated functions which are constant multiples, sums, products and/or quotients of much simpler functions. To achieve this, we can make use of our knowledge concerning the derivatives of these simpler functions. We have four extremely useful results.

CONSTANT MULTIPLE RULE. *Suppose that $m(x) = cf(x)$, where c is a fixed real number. Then*

$$m'(x) = cf'(x)$$

for every real number x for which $f'(x)$ exists.

SUM RULE. *Suppose that $s(x) = f(x) + g(x)$ and $d(x) = f(x) - g(x)$. Then*

$$s'(x) = f'(x) + g'(x) \qquad \text{and} \qquad d'(x) = f'(x) - g'(x)$$

for every real number x for which $f'(x)$ and $g'(x)$ exist.

EXAMPLE 11.2.1. Consider the function $h(x) = 5x^2 + 3x^5$. We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = 5x^2$ and $g(x) = 3x^5$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Next, the function $f(x) = 5x^2$ is a constant (5) multiple of the function x^2, and so it follows from the constant multiple rule and the rule on the derivatives of powers that

$$f'(x) = 5(x^2)' = 5(2x) = 10x.$$

Similarly, the function $g(x) = 3x^5$ is a constant (3) multiple of the function x^5, and so it follows from the constant multiple rule and the rule on the derivatives of powers that

$$g'(x) = 3(x^5)' = 3(5x^4) = 15x^4.$$

Hence $h'(x) = 10x + 15x^4$.

EXAMPLE 11.2.2. Consider the function $h(x) = (3x)^4 - (2x)^6$. We can write

$$h(x) = f(x) - g(x),$$

where $f(x) = 81x^4$ and $g(x) = 64x^6$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 324x^3$ and $g'(x) = 384x^5$. Hence $h'(x) = 324x^3 - 384x^5$.

The sum rule can be extended to the sum or difference of more than two functions in the natural way. We illustrate the technique in the following three examples.

EXAMPLE 11.2.3. Consider the function $h(x) = 4x^3 - 15x^2 + 4x - 1$. We can write

$$h(x) = f(x) - g(x) + k(x) - t(x),$$

where $f(x) = 4x^3$, $g(x) = 15x^2$, $k(x) = 4x$ and $t(x) = 1$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) + k'(x) - t'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 12x^2$, $g'(x) = 30x$ and $k'(x) = 4$. Applying the rule on the derivatives of constants, we obtain $t'(x) = 0$. Hence $h'(x) = 12x^2 - 30x + 4$.

EXAMPLE 11.2.4. Consider the function $h(x) = 8x^3 - 2(x+2)^2 + 3$. Then $h(x) = 8x^3 - 2x^2 - 8x - 5$, and so we can write

$$h(x) = f(x) - g(x) - k(x) - t(x),$$

where $f(x) = 8x^3$, $g(x) = 2x^2$, $k(x) = 8x$ and $t(x) = 5$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) - k'(x) - t'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 24x^2$, $g'(x) = 4x$ and $k'(x) = 8$. Applying the rule on the derivatives of constants, we obtain $t'(x) = 0$. Hence $h'(x) = 24x^2 - 4x - 8$.

EXAMPLE 11.2.5. Consider the function $h(x) = (x^2 + 2x)^2$. Then $h(x) = x^4 + 4x^3 + 4x^2$, and so we can write

$$h(x) = f(x) + g(x) + k(x),$$

where $f(x) = x^4$, $g(x) = 4x^3$ and $k(x) = 4x^2$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x) + k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 4x^3$, $g'(x) = 12x^2$ and $k'(x) = 8x$. Hence $h'(x) = 4x^3 + 12x^2 + 8x$.

EXAMPLE 11.2.6. Consider the function

$$h(x) = \frac{3}{x} + 2x.$$

We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = 3x^{-1}$ and $g(x) = 2x$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = -3x^{-2}$ and $g'(x) = 2$. Hence $h'(x) = 2 - 3x^{-2}$.

EXAMPLE 11.2.7. Consider the function

$$h(x) = 6x^2\sqrt{x} - \frac{4}{\sqrt{x}} + 3x^{1/3}.$$

We can write

$$h(x) = f(x) - g(x) + k(x),$$

where $f(x) = 6x^{5/2}$, $g(x) = 4x^{-1/2}$ and $k(x) = 3x^{1/3}$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) + k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 15x^{3/2}$, $g'(x) = -2x^{-3/2}$ and $k'(x) = x^{-2/3}$. Hence $h'(x) = 15x^{3/2} + 2x^{-3/2} + x^{-2/3}$.

EXAMPLE 11.2.8. Consider the function $h(x) = \sqrt{3x} + \sqrt[3]{2x}$. We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = \sqrt{3}x^{1/2}$ and $g(x) = \sqrt[3]{2}x^{1/3}$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$$f'(x) = \frac{\sqrt{3}}{2}x^{-1/2} \qquad \text{and} \qquad g'(x) = \frac{\sqrt[3]{2}}{3}x^{-2/3}.$$

Hence

$$h'(x) = \frac{\sqrt{3}}{2}x^{-1/2} + \frac{\sqrt[3]{2}}{3}x^{-2/3} = \sqrt{\frac{3}{4x}} + \sqrt[3]{\frac{2}{27x^2}}.$$

PRODUCT RULE. *Suppose that $p(x) = f(x)g(x)$. Then*

$$p'(x) = f'(x)g(x) + f(x)g'(x)$$

for every real number x for which $f'(x)$ and $g'(x)$ exist.

EXAMPLE 11.2.9. Consider the function $h(x) = (x^3 - x^5)(x^2 + x^4)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^3 - x^5$ and $g(x) = x^2 + x^4$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain $f'(x) = 3x^2 - 5x^4$ and $g'(x) = 2x + 4x^3$. Hence

$$h'(x) = (3x^2 - 5x^4)(x^2 + x^4) + (x^3 - x^5)(2x + 4x^3) = 5x^4 - 9x^8.$$

Alternatively, we observe that $h(x) = (x^3 - x^5)(x^2 + x^4) = x^5 - x^9$. Applying the sum rule and the rule on the derivatives of powers, we obtain $h'(x) = 5x^4 - 9x^8$ as before.

EXAMPLE 11.2.10. Let us return to Example 11.2.5 and consider again the function $h(x) = (x^2+2x)^2$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = g(x) = x^2 + 2x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule, the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = g'(x) = 2x + 2$. Hence

$$h'(x) = (2x + 2)(x^2 + 2x) + (x^2 + 2x)(2x + 2) = 2(2x + 2)(x^2 + 2x) = 4x^3 + 12x^2 + 8x$$

as before. We shall return to example again in Section 12.1.

EXAMPLE 11.2.11. Consider the function $h(x) = (x^2 + x)(x^3 - 6x^2 + 2x)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^2 + x$ and $g(x) = x^3 - 6x^2 + 2x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule, the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 2x + 1$ and $g'(x) = 3x^2 - 12x + 2$. Hence

$$h'(x) = (2x + 1)(x^3 - 6x^2 + 2x) + (x^2 + x)(3x^2 - 12x + 2).$$

EXAMPLE 11.2.12. Consider the function

$$h(x) = (x + \sqrt{x})\left(x - \frac{1}{\sqrt{x}}\right).$$

We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x + x^{1/2}$ and $g(x) = x - x^{-1/2}$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain

$$f'(x) = 1 + \frac{1}{2}x^{-1/2} \qquad \text{and} \qquad g'(x) = 1 + \frac{1}{2}x^{-3/2}.$$

Hence

$$\begin{aligned} h'(x) &= \left(1 + \frac{1}{2}x^{-1/2}\right)(x - x^{-1/2}) + (x + x^{1/2})\left(1 + \frac{1}{2}x^{-3/2}\right) \\ &= \left(x - x^{-1/2} + \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1}\right) + \left(x + \frac{1}{2}x^{-1/2} + x^{1/2} + \frac{1}{2}x^{-1}\right) \\ &= 2x + \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}. \end{aligned}$$

Alternatively, we observe that $h(x) = (x + x^{1/2})(x - x^{-1/2}) = x^2 - x^{1/2} + x^{3/2} - 1$. Applying the sum rule and the rules on the derivatives of powers and constants, we obtain

$$h'(x) = 2x - \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2}$$

as before.

The product rule can be extended to the product of more than two functions. The extension is at first sight somewhat less obvious than in the case of the sum rule. However, with a bit of care, it is in fact rather straightforward.

EXAMPLE 11.2.13. Consider the function $h(x) = (x^2 + 4x)(2x + 1)(6 - 2x^2)$. We can write

$$h(x) = f(x)r(x),$$

where $f(x) = x^2 + 4x$ and $r(x) = (2x + 1)(6 - 2x^2)$. It follows from the product rule that

$$h'(x) = f'(x)r(x) + f(x)r'(x).$$

We can now write

$$r(x) = g(x)k(x),$$

where $g(x) = 2x + 1$ and $k(x) = 6 - 2x^2$. It follows from the product rule that

$$r'(x) = g'(x)k(x) + g(x)k'(x).$$

Hence $h(x) = f(x)g(x)k(x)$, and

$$h'(x) = f'(x)g(x)k(x) + f(x)g'(x)k(x) + f(x)g(x)k'(x).$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 2x + 4$, $g'(x) = 2$ and $k'(x) = -4x$. Hence

$$h'(x) = (2x + 4)(2x + 1)(6 - 2x^2) + 2(x^2 + 4x)(6 - 2x^2) - 4x(x^2 + 4x)(2x + 1).$$

REMARK. The interested reader is challenged to show that if $p(x) = f(x)g(x)k(x)t(x)$, then

$$p'(x) = f'(x)g(x)k(x)t(x) + f(x)g'(x)k(x)t(x) + f(x)g(x)k'(x)t(x) + f(x)g(x)k(x)t'(x).$$

QUOTIENT RULE. *Suppose that $q(x) = f(x)/g(x)$. Then*

$$q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

for every real number x for which $f'(x)$ and $g'(x)$ exist, and for which $g(x) \neq 0$.

EXAMPLE 11.2.14. Consider the function

$$h(x) = \frac{x^2 - 1}{x^3 + 2x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = x^2 - 1$ and $g(x) = x^3 + 2x$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 2x$ and $g'(x) = 3x^2 + 2$. Hence

$$h'(x) = \frac{2x(x^3 + 2x) - (x^2 - 1)(3x^2 + 2)}{(x^3 + 2x)^2}.$$

Example 11.2.15. Consider the function

$$h(x)=\frac{4x^2+1}{3x}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=4x^2+1$ and $g(x)=3x$. It follows from the quotient rule that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x)=8x$ and $g'(x)=3$. Hence

$$h'(x)=\frac{24x^2-3(4x^2+1)}{9x^2}.$$

Example 11.2.16. Consider the function

$$h(x)=\frac{3x^2+4x^7}{5x^{-2}+3}.$$

We can write

$$h(x)=\frac{f(x)}{g(x)},$$

where $f(x)=3x^2+4x^7$ and $g(x)=5x^{-2}+3$. It follows from the quotient rule that

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x)=6x+28x^6$ and $g'(x)=-10x^{-3}$. Hence

$$\begin{aligned}h'(x)&=\frac{(5x^{-2}+3)(6x+28x^6)+10x^{-3}(3x^2+4x^7)}{(5x^{-2}+3)^2}=\frac{30x^{-1}+140x^4+18x+84x^6+30x^{-1}+40x^4}{25x^{-4}+30x^{-2}+9}\\&=\frac{60x^{-1}+18x+180x^4+84x^6}{25x^{-4}+30x^{-2}+9}\times\frac{x^4}{x^4}=\frac{60x^3+18x^5+180x^8+84x^{10}}{25+30x^2+9x^4}.\end{aligned}$$

Alternatively, we observe that

$$h(x)=\frac{3x^2+4x^7}{5x^{-2}+3}\times\frac{x^2}{x^2}=\frac{3x^4+4x^9}{5+3x^2}.$$

We can write

$$h(x)=\frac{k(x)}{t(x)},$$

where $k(x)=3x^4+4x^9$ and $t(x)=5+3x^2$. It follows from the quotient rule that

$$h'(x)=\frac{t(x)k'(x)-k(x)t'(x)}{t^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x)=12x^3+36x^8$ and $g'(x)=6x$. Hence

$$\begin{aligned}h'(x)&=\frac{(5+3x^2)(12x^3+36x^8)-6x(3x^4+4x^9)}{(5+3x^2)^2}=\frac{60x^3+36x^5+180x^8+108x^{10}-18x^5-24x^{10}}{25+30x^2+9x^4}\\&=\frac{60x^3+18x^5+180x^8+84x^{10}}{25+30x^2+9x^4}\end{aligned}$$

as before.

EXAMPLE 11.2.17. Consider the function

$$h(x) = \frac{(x^2+4)(x-2)}{x^2+2}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = (x^2+4)(x-2)$ and $g(x) = x^2+2$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

We can now write

$$f(x) = k(x)t(x),$$

where $k(x) = x^2+4$ and $t(x) = x-2$. It follows from the product rule that

$$f'(x) = k'(x)t(x) + k(x)t'(x).$$

Hence

$$h(x) = \frac{k(x)t(x)}{g(x)},$$

and

$$h'(x) = \frac{g(x)k'(x)t(x) + g(x)k(x)t'(x) - k(x)t(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x) = 2x$, $t'(x) = 1$ and $g'(x) = 2x$. Hence

$$h'(x) = \frac{2x(x^2+2)(x-2) + (x^2+2)(x^2+4) - 2x(x^2+4)(x-2)}{(x^2+2)^2} = \frac{x^4+2x^2+8x+8}{(x^2+2)^2}.$$

Alternatively, we observe that

$$f(x) = (x^2+4)(x-2) = x^3 - 2x^2 + 4x - 8.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 3x^2 - 4x + 4$. Hence

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} = \frac{(x^2+2)(3x^2-4x+4) - 2x(x^3-2x^2+4x-8)}{(x^2+2)^2} = \frac{x^4+2x^2+8x+8}{(x^2+2)^2}$$

as before.

For those who want a small challenge, here is one more example.

EXAMPLE 11.2.18. Consider the function

$$h(x) = \frac{(4x-3)(2x^2-3x)}{(2x+2)(x^3+6)}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = (4x-3)(2x^2-3x)$ and $g(x) = (2x+2)(x^3+6)$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

We can now write

$$f(x) = k(x)t(x) \qquad \text{and} \qquad g(x) = u(x)v(x),$$

where $k(x) = 4x - 3$, $t(x) = 2x^2 - 3x$, $u(x) = 2x + 2$ and $v(x) = x^3 + 6$. It follows from the product rule that

$$f'(x) = k'(x)t(x) + k(x)t'(x) \qquad \text{and} \qquad g'(x) = u'(x)v(x) + u(x)v'(x).$$

Hence

$$h(x) = \frac{k(x)t(x)}{u(x)v(x)},$$

and

$$\begin{aligned} h'(x) &= \frac{u(x)v(x)k'(x)t(x) + u(x)v(x)k(x)t'(x) - k(x)t(x)u'(x)v(x) - k(x)t(x)u(x)v'(x)}{u^2(x)v^2(x)} \\ &= \frac{u(x)v(x)k'(x)t(x) + u(x)v(x)k(x)t'(x)}{u^2(x)v^2(x)} - \frac{k(x)t(x)u'(x)v(x) + k(x)t(x)u(x)v'(x)}{u^2(x)v^2(x)}. \end{aligned}$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x) = 4$, $t'(x) = 4x - 3$, $u'(x) = 2$ and $v'(x) = 3x^2$. Hence

$$\begin{aligned} h'(x) = {} & \frac{4(2x+2)(x^3+6)(2x^2-3x) + (2x+2)(x^3+6)(4x-3)^2}{(2x+2)^2(x^3+6)^2} \\ & - \frac{2(4x-3)(2x^2-3x)(x^3+6) + 3x^2(4x-3)(2x^2-3x)(2x+2)}{(2x+2)^2(x^3+6)^2}. \end{aligned}$$

11.3. Derivatives of the Trigonometric Functions

Consider the curve $y = f(x) = \sin x$. Suppose that $P(x, f(x))$ is a point on this curve. Consider another point $Q(x + h, f(x + h))$, where $h \neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points P and Q is equal to

$$\frac{f(x+h) - f(x)}{(x+h) - x} = \frac{\sin(x+h) - \sin x}{h}.$$

Consider the curve $y = g(x) = \cos x$. Suppose that $R(x, g(x))$ is a point on this curve. Consider another point $S(x + h, g(x + h))$, where $h \neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points R and S is equal to

$$\frac{g(x+h) - g(x)}{(x+h) - x} = \frac{\cos(x+h) - \cos x}{h}.$$

We now move the point Q along the curve $y = f(x) = \sin x$ towards the point P, and move the point S along the curve $y = g(x) = \cos x$ towards the point R. Recall Example 3.3.9, that when h is very close to 0, we have

$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x \qquad \text{and} \qquad \frac{\cos(x+h) - \cos x}{h} \approx -\sin x.$$

We have established the first two parts of the result below.

DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS.

(a) If $f(x) = \sin x$, then $f'(x) = \cos x$.
(b) If $g(x) = \cos x$, then $g'(x) = -\sin x$.
(c) If $t(x) = \tan x$, then $t'(x) = \sec^2 x$.
(d) If $t(x) = \cot x$, then $t'(x) = -\csc^2 x$.
(e) If $t(x) = \sec x$, then $t'(x) = \tan x \sec x$.
(f) If $t(x) = \csc x$, then $t'(x) = -\cot x \csc x$.

PROOF. The proofs of parts (c)–(f) depend on the quotient rule as well as parts (a) and (b). For the sake of convenience, we use the functions $f(x) = \sin x$ and $g(x) = \cos x$ throughout this proof, as well as the function $c(x) = 1$, with $c'(x) = 0$.

(c) Suppose that $t(x) = \tan x$. Then $t(x) = f(x)/g(x)$. It follows from the quotient rule that

$$t'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

(d) Suppose that $t(x) = \cot x$. Then $t(x) = g(x)/f(x)$. It follows from the quotient rule that

$$t'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{f^2(x)} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

(e) Suppose that $t(x) = \sec x$. Then $t(x) = c(x)/g(x)$. It follows from the quotient rule that

$$t'(x) = \frac{g(x)c'(x) - c(x)g'(x)}{g^2(x)} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \tan x \sec x.$$

(f) Suppose that $t(x) = \csc x$. Then $t(x) = c(x)/f(x)$. It follows from the quotient rule that

$$t'(x) = \frac{f(x)c'(x) - c(x)f'(x)}{f^2(x)} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \times \frac{1}{\sin x} = -\cot x \csc x. \quad \clubsuit$$

We next combine our knowledge on trigonometric functions with the arithmetic of derivatives. The reader is advised to identify the rules used at each step in the following examples.

EXAMPLE 11.3.1. Consider the function $h(x) = (x^3 - 2)(\sin x + \cos x)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^3 - 2$ and $g(x) = \sin x + \cos x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Observe next that $f'(x) = 3x^2$ and $g'(x) = \cos x - \sin x$. Hence

$$h'(x) = 3x^2(\sin x + \cos x) + (x^3 - 2)(\cos x - \sin x).$$

EXAMPLE 11.3.2. Consider the function

$$h(x) = \frac{\sin x}{x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = \sin x$ and $g(x) = x$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = \cos x$ and $g'(x) = 1$. Hence

$$h'(x) = \frac{x \cos x - \sin x}{x^2}.$$

EXAMPLE 11.3.3. Consider the function $h(x) = \sin^2 x$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = g(x) = \sin x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Observe next that $f'(x) = g'(x) = \cos x$. Hence

$$h'(x) = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x.$$

EXAMPLE 11.3.4. Consider the function $y = \sin 2x$. We can write

$$h(x) = 2f(x)g(x),$$

where $f(x) = \sin x$ and $g(x) = \cos x$. It follows that

$$h'(x) = 2(f'(x)g(x) + f(x)g'(x)).$$

Observe next that $f'(x) = \cos x$ and $g'(x) = -\sin x$. Hence

$$h'(x) = 2(\cos^2 x - \sin^2 x) = 2 \cos 2x.$$

We shall return to Examples 11.3.3 and 11.3.4 in Section 12.1.

EXAMPLE 11.3.5. Consider the function

$$h(x) = \frac{\cos x}{x^2 - x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = \cos x$ and $g(x) = x^2 - x$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = -\sin x$ and $g'(x) = 2x - 1$. Hence

$$h'(x) = \frac{(x - x^2) \sin x - (2x - 1) \cos x}{(x^2 - x)^2}.$$

EXAMPLE 11.3.6. Consider the function

$$h(x) = \frac{\sin x + \cos x}{1 - x^4}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = \sin x + \cos x$ and $g(x) = 1 - x^4$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = \cos x - \sin x$ and $g'(x) = -4x^3$. Hence

$$h'(x) = \frac{(1-x^4)(\cos x - \sin x) + 4x^3(\sin x + \cos x)}{(1-x^4)^2}.$$

EXAMPLE 11.3.7. Consider the function

$$h(x) = \left(\frac{x^2+1}{\cos x}\right)\sin x.$$

We can write

$$h(x) = f(x)g(x),$$

where

$$f(x) = \frac{x^2+1}{\cos x} \qquad \text{and} \qquad g(x) = \sin x.$$

It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

We can also write

$$f(x) = \frac{k(x)}{t(x)},$$

where $k(x) = x^2 + 1$ and $t(x) = \cos x$. It follows that

$$f'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)},$$

and so

$$h'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)}g(x) + \frac{k(x)}{t(x)}g'(x).$$

Observe next that $k'(x) = 2x$, $t'(x) = -\sin x$ and $g'(x) = \cos x$. Hence

$$\begin{aligned} h'(x) &= \left(\frac{2x\cos x + (x^2+1)\sin x}{\cos^2 x}\right)\sin x + \left(\frac{x^2+1}{\cos x}\right)\cos x \\ &= 2x\tan x + (x^2+1)\tan^2 x + (x^2+1) = 2x\tan x + (x^2+1)\sec^2 x. \end{aligned}$$

Alternatively, we observe that $h(x) = (x^2+1)\tan x$. We can write

$$h(x) = u(x)v(x),$$

where $u(x) = x^2 + 1$ and $v(x) = \tan x$. It follows that

$$h'(x) = u'(x)v(x) + u(x)v'(x).$$

Observe next that $u'(x) = 2x$ and $v'(x) = \sec^2 x$. Hence $h'(x) = 2x\tan x + (x^2+1)\sec^2 x$ as before.

EXAMPLE 11.3.8. Consider the function $h(x) = \sin^2 x + \cos^2 x$. We can write

$$h(x) = f(x)g(x) + k(x)t(x),$$

where $f(x) = g(x) = \sin x$ and $k(x) = t(x) = \cos x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x) + u'(x)v(x) + u(x)v'(x).$$

Observe next that $f'(x) = g'(x) = \cos x$ and $k'(x) = t'(x) = -\sin x$. Hence

$$h'(x) = \cos x \sin x + \sin x \cos x - \sin x \cos x - \cos x \sin x = 0.$$

A far simpler way to obtain the same result is to merely observe that $h(x) = 1$.

Problems for Chapter 11

1. For each of the following functions $f(x)$, write down the derivative $f'(x)$ as a function of x, and find the slope of the tangent at the point $P(1, f(1))$:

   a) $f(x) = x^4$ b) $f(x) = 5x^2$ c) $f(x) = \frac{1}{6}x^{-3}$ d) $f(x) = \pi x^{1.5}$

2. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants and sums:

   a) $h(x) = 6x^3$ b) $h(x) = 5x^{-7}$
   c) $h(x) = 12x - 3x^2$ d) $h(x) = x^3 + 4x$
   e) $h(x) = 6x^2 - 40x$ f) $h(x) = x^7 + 6x^5 - 8x^2 + 3x$
   g) $h(x) = -\dfrac{3}{x}$ h) $h(x) = \dfrac{7}{x^6}$
   i) $h(x) = \dfrac{6}{x^2}$ j) $h(x) = x^3 + 3x - \dfrac{5}{x^3}$
   k) $h(x) = x^2 - 10x + 100 + \dfrac{4}{x}$ l) $h(x) = x^{100} + 50x + 1 - 2x^{-3} + 7x^{-6}$
   m) $h(x) = \pi x^3 - \dfrac{\pi^2}{x^6}$ n) $h(x) = x^2(x^3 + 3x)$
   o) $h(x) = (x^2 + 3)(2x - 5)$ p) $h(x) = -5\sqrt{x}$
   q) $h(x) = \dfrac{\sqrt{x}}{x^3}$ r) $h(x) = \sqrt{3x}$
   s) $h(x) = \sqrt{4x} + \sqrt{\dfrac{4}{x}}$ t) $h(x) = x^5 + 6x^{-3/2}$

3. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums and products as appropriate:

   a) $h(x) = (x^2 + 3)(2x - 5)$ b) $h(x) = (x^2 - x + 2)(x^2 - 2)$
   c) $h(x) = (x^2 + 5)(x^3 - 4x^2)$ d) $h(x) = (x^4 - 3x^3 + 2x)(3x^2 + 4x)$
   e) $h(x) = (x^9 + 2x^3)x^{-4}$ f) $h(x) = (x^4 - 2x^3 + 7x + 8)^2$
   g) $h(x) = x^{2/3}(x + 2)$ h) $h(x) = (x + 3)(x - 5)(x^2 - 4)$
   i) $h(x) = x^{1/2}(x^3 + x - 2)(3x + 1)$ j) $h(x) = x(x - 1)(x - 2)$

4. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums, products and quotients as appropriate:

   a) $h(x) = \dfrac{1}{x^4 + x^3 + 1}$ b) $h(x) = 1 + \dfrac{3}{x} - \dfrac{2}{x^2}$ c) $h(x) = \dfrac{x - 2}{x + 1}$
   d) $h(x) = \dfrac{1 + x^2}{1 - x^2}$ e) $h(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$ f) $h(x) = \dfrac{x}{x + x^{-1}}$
   g) $h(x) = \dfrac{2x + 3}{3x + 2}$ h) $h(x) = \dfrac{2x + 1}{x - 1}$

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