

For any real number x the absolute value or modulus of x is denoted by $|x|$ (read "the absolute value of x ") and is defined as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

As can be seen from the above definition, the absolute value of a number is always either positive or zero, but never negative. The vertical lines in the symbol $|x|$ are called absolute value bars.

From an analytic geometry point of view, the absolute value of a real number is that number's distance from zero along the real number line (it tells us how far it is from 0 to x but not the direction). More generally the absolute value of the difference of two real numbers is the distance between them.



$|a - b|$ is the difference between the larger and the smaller of the two numbers. In other words, $|a - b|$ is the distance between the points a and b (as illustrated above).

$$\sqrt{x^2} = |x|$$

The notation $|a|$ was introduced by Karl Weierstrass in 1841.

ARITHMETIC WITH ABSOLUTE VALUES

Let $x, y \in \mathbb{R}$.

$$|x| \geq 0, \quad |x| = |-x|, \quad |x \cdot y| = |x| \cdot |y|, \quad |x + y| \leq |x| + |y|$$

RELATION BETWEEN INTERVALS AND ABSOLUTE VALUES

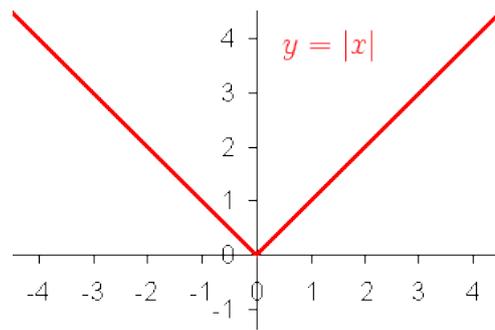
If D is any positive number, then

- $|x| < D \Leftrightarrow -D < x < D$ - the interval $(-D, D)$
- $|x| \leq D \Leftrightarrow -D \leq x \leq D$ - the interval $[-D, D]$
- $|x| > D \Leftrightarrow x < -D$ or $x > D$ - the intervals $(-\infty, -D) \cup (D, \infty)$
- $|x| \geq D \Leftrightarrow x \leq -D$ or $x \geq D$ - the intervals $(-\infty, -D] \cup [D, \infty)$

THE ABSOLUTE VALUE FUNCTION

The absolute value function

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



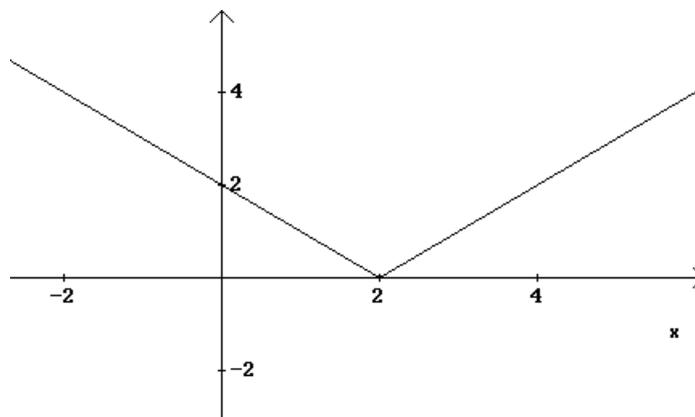
is monotonically decreasing on the interval $(-\infty, 0]$ and monotonically increasing on the interval $[0, \infty)$. Since a real number and its negative have the same absolute value, it is an even function, and is hence not invertible. The real absolute value function is continuous everywhere. It is differentiable everywhere except for $x = 0$ (since it has a "sharp corner"). The derivative of the real absolute value function is the signum function, $\text{sgn}(x)$, which is defined as

$$\text{sgn}(x) = \frac{x}{|x|}$$

for $x \neq 0$.

EXAMPLE

a) $f(x) = |x - 2|$



b) $f(x) = |(x - 2)^2 - 4|$

