Author: Piotrowski Paweł
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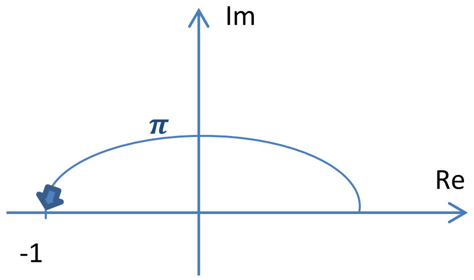
How to calculate complex roots

Example: $\sqrt[3]{-1}$ a cubic root of -1

$z = -1$ my number **z** is equal to -1 (or -1 +0i)

$z = -1 + 0i$ I will draw **z** to find its angle



$\varphi = \pi$ (= 180˚)

Now we can calculate the modulus of **z**:

$$|z| = \sqrt{(-1)^2 + 0^2} = 1$$

I expect to find three roots: z_0, z_1, z_2

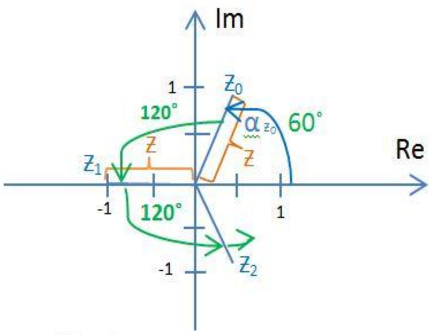
z_0 can be calculated from formula :

$$z_0 = \sqrt[3]{|z|} \cdot \left(\cos\frac{\varphi + 2\pi \cdot 0}{3} + i \cdot \sin\frac{\varphi + 2\pi \cdot 0}{3}\right)$$

Because we expect to find 3 roots

$$z_0 = \sqrt[3]{1} \cdot \left(\cos\frac{\pi + 2\pi \cdot 0}{3} + i \sin\frac{\pi + 2\pi \cdot 0}{3}\right) = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{2} + \underbrace{\frac{\sqrt{3}}{2}}_{\approx 0,8660\ldots}i$$

Now I will draw **z_0** and find its angle, then I will try to "guess" other roots.



$\alpha_{z_0} = ?$

$$\tan(\alpha_{z_0}) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\alpha_{z_0} = 60^\circ = \frac{\pi}{3}$$

I will have 3 roots which will divide the 360˚ into 3 identical parts:

$$\frac{360^\circ}{3} = 120^\circ$$

So each part will have 120˚.

z_1 is on the horizontal "Re" axis. And

$$z = |z_0| = |z_1|$$

$$z = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$z_1 = -1$

The last root lies 120˚ away from **z_0** counterclockwise and it's easy to see that it's symmetrical to z_0

$$z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Answer: $\sqrt[3]{-1} \in \left\{\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -1\right\}$

Check

$$z_0^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 =$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{4} + \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2\right) =$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{4} - \frac{3}{4} + \frac{\sqrt{3}}{2}i\right) =$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2}i - \frac{1}{2}\right) =$$

$$= \frac{\sqrt{3}}{4}i - \frac{1}{4} + \frac{3}{4}i^2 - \frac{\sqrt{3}}{4}i = -1$$

$$z_1^3 = (-1)^3 = -1$$

$$z_2^3 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2 =$$

$$= \left(\frac{1}{2} - \sqrt{\frac{3}{2}}i\right)\left(\frac{1}{4} - \frac{\sqrt{3}}{2}i - \frac{3}{4}\right) =$$

$$= \left(\frac{1}{2} - \sqrt{\frac{3}{2}}i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$$

$$= -\frac{1}{4} - \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i - \frac{3}{4} = -1$$