How to calculate complex roots

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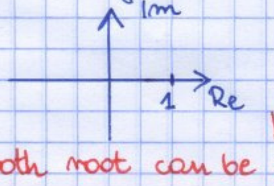
$\sqrt[8]{1}$ an eighth root of 1.

$z = 1$ my number z is equal to 1 (or I can write: $1+0i$)

Now, I have to draw my number to find its angle.



It's easy to notice that: $\varphi = 0$

I have to calculate the modulus of z: $|z| = \sqrt{1^2} = 1$

I expect to find eight roots: $z_0, z_1, z_2, z_3, z_4, z_5, z_6$ and z_7

The zeroth root can be calculated from formula:

$$z_0 = \sqrt[8]{|z|}\left(\cos\frac{\varphi + 2\pi \cdot 0}{8} + i\sin\frac{\varphi + 2\pi \cdot 0}{8}\right)$$

because I expect to find 8 roots

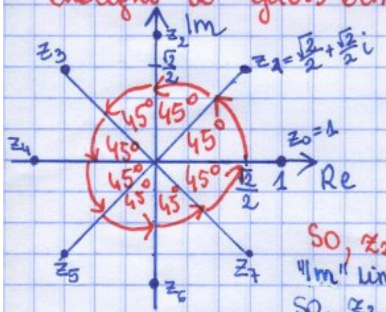
because it's the root number 0

so $z_0 = \sqrt[8]{1}\cdot\left(\cos\frac{0+2\pi\cdot 0}{8} + i\sin\frac{0+2\pi\cdot 0}{8}\right) = 1\,(\underbrace{\cos 0}_{1} + \underbrace{i\sin 0}_{0}) = 1$

Unfortunately it's not enough, to be able to "guess" other roots, so I have to calculate the next root:

$z_1 = \sqrt[8]{1}\cdot\left(\cos\frac{0+2\pi\cdot 1}{8} + i\sin\frac{0+2\pi\cdot 1}{8}\right) = 1\left(\underbrace{\cos\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} + \underbrace{i\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

I will draw z_0 and z_1 and find the angle between them. Maybe it will be enought to "guess" other roots.



I will have 8 roots and they divide circle into 8 identical parts.

$$\frac{360°}{8} = 45°$$

Each part lasts for 45°

So, z_2 must lie somewhere on the horizontal "Im" line. I know that $\rho = |z_0| = |z_1| = |z_2| \ldots$

So, $z_2 = 1i$

z_3 lies 45° away from z_2 and it's easy to see that it's symetrical to z_1 so, $z_3 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

z_4 lies on "Re" line, and it is symetrical to z_0, so, $z_4 = -1$

z_5 lies 45° away from z_4 and it is symetrical to z_3; so, $z_5 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

z_6 must lie on "Im" line and it is symetrical to z_2 so, $z_6 = -1i$

z_7 lies 45° away from z_6 and it's symetrical to z_1 so $z_7 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

Let's check:

$z_0^8 = 1$ ✓

$z_1^8 = \left(\left(\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2\right)^2\right)^2 = \left(\left(\frac{2}{4} + 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}i + \frac{2}{4}i^2\right)^2\right)^2 = \left(\left(\frac{1}{2} + i - \frac{1}{2}\right)^2\right)^2 = (i^2)^2 = 1$ ✓

$z_2^8 = 1^8 = 1$

$z_3^8 = \left(\left(\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2\right)^2\right)^2 = \left(\left(\frac{1}{2} + i - \frac{1}{2}\right)^2\right)^2 = (i^2)^2 = (-1)^2 = 1$ ✓

$z_4^8 = (-1)^8 = 1$ ✓

$z_5 = \left(\left(\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2\right)^2\right)^2 = \left(\left(\frac{2}{4} - 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}i + \frac{2}{4}i^2\right)^2\right)^2 = (i^2)^2 = 1$ ✓

$z_6 = (-i)^8 = 1$ ✓

$z_7 = \left(\left(\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2\right)^2\right)^2 = \left(\left(\frac{1}{2} - 2\cdot\frac{\sqrt{2}}{2}i + \frac{1\cdot 2}{2}i\right)^2\right)^2 = ((-i)^2)^2 = 1$ ✓

Answer: $\sqrt[8]{1} \in \left\{\pm 1, \pm\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \pm i, \pm\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right\}$