# How to invert matrices using Gaussian elemination algorithm.

Example: $A = \frac{1}{5} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

1° I need to multiply the matrix by $\frac{1}{5}$ to not have factor in front of matrix. Then, I need to rewrite the matrix, then follow it by vertical line and a unit matrix with appropriate dimensions - in my case - $J_4$.

$$\left[\begin{array}{cccc|cccc} 2/5 & -1/5 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1/5 & 2/5 & -1/5 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1/5 & 2/5 & -1/5 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1/5 & 2/5 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{(right block: } \overbrace{\phantom{1000}}^{J_4}\text{)}$$

2° The goal is to perform a certain number of operations that will produce a matrix: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & A^{-1} \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right]$

$r_1 \to r_1 \cdot 5$, $r_2 \to r_2 \cdot 5$, $r_3 \to r_3 \cdot 5$, $r_4 \to r_4 \cdot 5$

The first step is to multiply every row by 5 to not have fractions inside the matrix.

$$\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 5 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 5 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 5 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \end{array}\right] \xrightarrow{r_1 \Leftarrow r_2} \left[\begin{array}{cccc|cccc} -1 & 2 & -1 & 0 & 0 & 5 & 0 & 0 \\ 2 & -1 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 5 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \end{array}\right] \xrightarrow{r_2 \to r_2 + 2r_1} \left[\begin{array}{cccc|cccc} -1 & 2 & -1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 3 & -2 & 0 & 5 & 10 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 5 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \end{array}\right]$$

It's good to have 1 in the upper left corner. I choose to exchange 1st and 2nd row to not produce fractions dividing 1st row.

If I get rid of this number my 1st column will look like in $J_4$!

I would like to have zero here. And here to have 1.

$r_1 \to r_1 + 2r_3$

$r_2 \to r_2 + 2r_3$

$$\left[\begin{array}{cccc|cccc} -1 & 0 & 3 & -2 & 0 & 5 & 10 & 0 \\ 0 & 1 & 2 & -2 & 5 & 10 & 10 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 5 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \end{array}\right] \xrightarrow{r_3 \to r_3 + r_2} \left[\begin{array}{cccc|cccc} -1 & 0 & 3 & -2 & 0 & 5 & 10 & 0 \\ 0 & 1 & 2 & -2 & 5 & 10 & 10 & 0 \\ 0 & 0 & 4 & -3 & 5 & 10 & 15 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \end{array}\right] \xrightarrow[r_2 \to r_2 + 2r_4]{r_1 \to r_1 + 3r_4} \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 4 & 0 & 5 & 10 & 15 \\ 0 & 1 & 0 & 2 & 5 & 10 & 10 & 10 \\ 0 & 0 & 4 & -3 & 5 & 10 & 15 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \end{array}\right] \xrightarrow{r_3 \Leftrightarrow r_4} \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 4 & 0 & 5 & 10 & 15 \\ 0 & 1 & 0 & 2 & 5 & 10 & 10 & 10 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \\ 0 & 0 & 4 & -3 & 5 & 10 & 15 & 0 \end{array}\right]$$

I would like to have 0 here.

I'll get rid of these numbers.

I would like to have 1 in 3rd row, so I exchange 3rd and 4th.

Now, I must get rid of this number.

$r_4 \to r_4 + 4r_3$

$$\left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 4 & 0 & 5 & 10 & 15 \\ 0 & 1 & 0 & 2 & 5 & 10 & 10 & 10 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 5 & 10 & 15 & 20 \end{array}\right] \xrightarrow{r_4 \to r_4/5} \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 4 & 0 & 5 & 10 & 15 \\ 0 & 1 & 0 & 2 & 5 & 10 & 10 & 10 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right] \xrightarrow[\substack{r_2 \to r_2 - 2r_4 \\ r_3 \to r_3 - 2r_4}]{r_1 \to r_1 - 4r_4} \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & -4 & -3 & -2 & -1 \\ 0 & 1 & 0 & 0 & 3 & 6 & 4 & 2 \\ 0 & 0 & -1 & 0 & -2 & -4 & -6 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right]$$

I would like to have 1 here.

If I get rid of these number I will have a unit matrix!

The last step is to get rid of minuses

$r_1 \to r_1 \cdot (-1)$

$r_3 \to r_3 \cdot (-1)$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 & 6 & 4 & 2 \\ 0 & 0 & 1 & 0 & 2 & 4 & 6 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right]$$

In the end my inverse matrix looks like:

$$\left[ \; J_4 \; \underbrace{\begin{vmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}}_{\Uparrow \; A^{-1}} \right]$$

Operations which I was using:

1. $r_i \to r_j$
2. $r_i \to r_i \cdot c$ ; $r_i \to r_i / c$
3. $r_i \to r_i \pm d \cdot r_j$
4. $r_i \Leftrightarrow r_j$
5. $r_i \to r_i \pm r_j$

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