How to use Gaussian Elimination to solve a system of equations

- Example

$$\begin{cases} -2x + 3y + z + 4k = 1 \\ x - y + 2z + 3k = 2 \\ 2x + y + z - 2k = 3 \end{cases}$$

- I can rewrite this as three matrices:

$$A = \begin{bmatrix} -2 & 3 & 2 & 4 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \\ z \\ k \end{bmatrix}$$

- In the next step I have to transform matrix $[A \mid B]$ into $[I_3 \mid \begin{smallmatrix} x \\ y \\ z \end{smallmatrix}]$ using elementary operations on matrices:

$$\left|\begin{array}{cccc|c} -2 & 3 & 1 & 4 & 1 \\ 1 & -1 & 2 & 3 & 2 \\ 2 & 1 & 1 & -2 & 3 \end{array}\right| \xrightarrow{r_1 \leftrightarrow r_2} \left|\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ -2 & 3 & 1 & 4 & 1 \\ 2 & 1 & 1 & -2 & 3 \end{array}\right| \xrightarrow[r_3 : r_3 - 2r_1]{r_2 : r_2 + 2r_1} \left|\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 10 & 5 \\ 0 & 3 & -3 & -8 & -1 \end{array}\right| \xrightarrow{r_1 : r_1 + r_2} \left|\begin{array}{cccc|c} 1 & 0 & 7 & 13 & 7 \\ 0 & 1 & 5 & 10 & 5 \\ 0 & 3 & -3 & -8 & -1 \end{array}\right| \xrightarrow{r_3 : r_3 - 3r_2}$$

1° I want to have 1 in here.

2° I need 0's in here.

3° I want 0 in here.

4° I need 0 in here

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 7 & 13 & 7 \\ 0 & 1 & 5 & 10 & 5 \\ 0 & 0 & -18 & -38 & -16 \end{array}\right] \xrightarrow{r_3 : r_3 : (-18)} \left[\begin{array}{cccc|c} 1 & 0 & 7 & 13 & 7 \\ 0 & 1 & 5 & 10 & 5 \\ 0 & 0 & 1 & 2\frac{1}{9} & \frac{8}{9} \end{array}\right] \xrightarrow{r_1 : r_1 - 7r_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1\frac{7}{9} & \frac{7}{9} \\ 0 & 1 & 5 & 10 & 5 \\ 0 & 0 & 1 & 2\frac{1}{9} & \frac{8}{9} \end{array}\right] \xrightarrow{r_2 : r_2 - 5r_3} \begin{array}{c} \begin{array}{cccc} x & y & z & k \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1\frac{7}{9} & \frac{7}{9} \\ 0 & 1 & 0 & -\frac{5}{9} & \frac{5}{9} \\ 0 & 0 & 1 & 2\frac{1}{9} & \frac{8}{9} \end{array}\right] \end{array}$$

5° To have last row I need 1 in here.

6° To end 1st row I need 0 in here

7° To have I_3 I need 0 in here

I_3

One extra column.

Result:

- $$\begin{cases} x - 1\frac{7}{9}k = \frac{7}{9} \\ y - \frac{5}{9}k = \frac{5}{9} \\ z + 2\frac{1}{9}k = \frac{8}{9} \end{cases} \Rightarrow \begin{cases} x = 1\frac{7}{9}k + \frac{7}{9} \\ y = \frac{5}{9}k + \frac{5}{9} \\ z = -2\frac{1}{9}k + \frac{8}{9} \end{cases}$$