Two-variable functions

Exercise 1. Try out the commands that draw sets of points satisfying certain conditions:

Plot[x^2+y^2<3]
Plot[x^2+y^2<3, x>0]
Plot[x^2+y^2<3, x>0,y>x]
Plot[x^2+y^2<3, x>0,y>=0,y<=x] – too many conditions, this instruction shouldn't work
Plot[y>=Sqrt[x]]
plot y>=Sqrt[x],x=-1..10 – a graph in a given interval

Draw sets of points satisfying the following conditions:

$$(a)\ 1 \le x^2 + y^2 < 3 \qquad (b)\ (x-2)^2 + (y-1)^2 \le 4$$

Exercise 2. Try out commands finding the limit of a sequence:

Limit (n,1/n) as n->Infinity
Limit (Cos[Pi*n],2^n) as n->Infinity – the interval given as an answer (*-1 to 1*) tells us that the limit doesn't exist
Limit (Atan[n],1-1/n^3) as n->Infinity

Find sequence limits:

$$(a)\ \left(\sqrt[n]{n}, \tfrac{n}{n+1}\right) \qquad (b)\ \left(\left(1+\tfrac{1}{n}\right)^n, \left(\tfrac{2}{3}\right)^n\right) \qquad (c)\ \left(\left(\tfrac{6}{5}\right)^n, \cos n\pi\right)$$

Exercise 3. This limit doesn't exist:

Limit x/y as x->0, y->0 - in return, we get the following information:
(limit does not exist)
(value depends on x,y path) – the value of the limit depends on the direction from which we get closer to the (0,0) point

This instruction will work exactly the same:
Limit x/y as (x,y)->(0,0)

Check that the following limits do not exist:

$$(a)\ \lim_{(x,y)\to(0,0)} \frac{x}{x+y} \qquad (b)\ \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \qquad (c)\ \lim_{(x,y)\to(0,1)} \frac{x^6}{y^3-1}$$

$$(d)\ \lim_{(x,y)\to(0,0)} \frac{\sin x}{\sin y} \qquad (e)\ \lim_{(x,y)\to(0,0)} \frac{1}{x+y} \qquad (f)\ \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Exercise 4. Sometimes it is worth to check how the graph of the function behaves in the surroundings of a given point. For example, when the limit doesn't exist:

Plot3D[x/y] – this instruction shows the graph of a function on a rather large domain
Plot3D[x/y,x=-0.1..0.1,y=-0.1..0.1] – this instruction shows the graph of a function on the set [-0.1,0.1] × [-0.1,0.1]

In the *Contour plot* area we can clearly see, that in the surrounding of the point (0,0) the function takes on different values depending on the direction from which we get closer to the (0,0) point.

A similar example: the **Limit x^3/(y^3-1) as (x,y)->(0,1)** doesn't exist and we can see the behavior of the function graph in the surrounding of the (0,1) points on a graph: **Plot3d[x^3/(y^3-1),x=-0.1..0.1,y=0.9..1.1]** in the *Contour plot* area.

Plot the graphs of functions from the previous exercise in the surrounding of a point given under the limit symbol.

Exercise 5. Try out the following commands:

D[x^2+3xy+4y^2,x] – partial derivative of the 1st degree with respect to x
D[x^2+3xy+4y^2,y] – partial derivative of the 1st degree with respect to y
D[x^2+3xy+4y^2,{x,2}] – partial derivative of the 2nd degree with respect only to x
D[x^2+3xy+4y^2,x,y] – partial derivative of the 2nd degree with respect to x and y

Find all partial derivatives of the 1st and the 2nd degree:

(a) $f(x,y) = x^2 + xy + y^2 - 3x^2 + 10x^4,$

(b) $f(x,y) = x^2 + xy + y^2,$

(c) $f(x,y) = x^y + y^x + 5,$

(d) $f(x,y) = 20x^2 - 10y^2x + 6xy + 12x^5y^{10},$

Exercise 6. The following commands find the minima, maxima and the saddle points of two-variable functions:

Minimize[(xy-3)^2+1,{x,y}]
Maximize[(xy-3)^2+1,{x,y}] – no maxima were found here
saddle points (xy-3)^2+1

Another example:

Minimize[x^3+3xy^2-51x-24y,{x,y}]
Maximize[x^3+3xy^2-51x-24y,{x,y}]
saddle points x^3+3xy^2-51x-24y

And another example:

Maximize[4xy-x^4-y^4,{x,y}] – two maxima were found
saddle points 4xy-x^4-y^4

Find minima, maxima and saddle points of the following functions:

a) $f(x,y) = x^3 + 3xy^2 - 51x - 24y,$

b) $f(x,y) = 3x^3 + 3x^2y - y^3 - 15x,$

c) $f(x,y) = x^3 + 3xy^2 - 15x - 12y,$

d) $f(x,y) = x^3 - 4xy + 2y^2,$

(*) Find extremes of the following function:

$$f(x,y,z) = x^2 + y^2 + z^2 - xy + x + 2z$$