CRAMER'S RULE

Find the value of variable x using Cramer's rule (do not use the Sarrus' method at any time)

$$\begin{cases} 3x + 7y + 2z + 4t = 0 \\ 2y + z = 0 \\ x + 4y + z = 1 \\ 5x + 3y + 2z = 0 \end{cases}$$

1) At the beginning we have to „transform" the system of equation into a matrix and we have:

$$\underbrace{\begin{bmatrix} x & y & z & t \\ 3 & 7 & 2 & 4 \\ 0 & 2 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 5 & 3 & 2 & 0 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

2) Then we have to calculate the determinant of matrix A

$$\det A = \begin{vmatrix} 3 & 7 & 2 & 4 \\ 0 & 2 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 5 & 3 & 2 & 0 \end{vmatrix} = 3 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 0 \end{vmatrix} + 7 \cdot (-1)^{1+2} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 2 & 0 \end{vmatrix} + 2 \cdot (-1)^{1+3} \begin{vmatrix} 0 & 2 & 0 \\ 1 & 4 & 0 \\ 5 & 3 & 0 \end{vmatrix} + 4 \cdot (-1)^{1+4} \begin{vmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 5 & 3 & 2 \end{vmatrix} =$$

$$= 3 \cdot \left(2 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} + 1 \cdot (-1)^{1+2} \begin{vmatrix} 4 & 0 \\ 3 & 0 \end{vmatrix}\right) - \underbrace{7 \left(1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 5 & 0 \end{vmatrix}\right)}_{0} + \underbrace{2 \cdot \left(2 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 5 & 0 \end{vmatrix}\right)}_{0} - 4 \cdot \left(2 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} + 1 \cdot (-1) \ldots\right.$$

$$= \underbrace{3 \cdot (2 \cdot 0 + (-1) \cdot 0)}_{0} - 0 + 0 - 4 \cdot (-2) \cdot (-3) + 1 \cdot (-17) = -24 - 17 = -41$$

$\det A = -41$

3) To calculate x, we also have to compute $\det A_x$.

To do that, we have to put the results from the right side of the equation instead of x-column and we have

$$\det A_x: \begin{vmatrix} 0 & 7 & 2 & 4 \\ 0 & 2 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{vmatrix} = 7 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} + 2 \cdot (-1)^{1+3} \begin{vmatrix} 0 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 3 & 0 \end{vmatrix} + 4 \cdot (-1)^{1+4} \begin{vmatrix} 0 & 2 & 1 \\ 1 & 4 & 1 \\ 0 & 3 & 2 \end{vmatrix} = \underbrace{-7 \cdot \left(1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\right)}_{0} + \underbrace{2 \cdot \left(2 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\right)}_{0} +$$

$$-4 \cdot \left(2 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1 \cdot (-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}\right) = 0 + 0 - 4 \cdot (-2) \cdot 2 + 1 \cdot 3 = 19$$

$$x: \frac{\det A_x}{\det A} = \frac{19}{-41} \Rightarrow x = -\frac{19}{41}$$

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CRAMER'S FORMULA → if dim $A = m \times n$ $(m=n)$ and $\det A \neq 0$

$$x_1 = \frac{\det A_1}{\det A}, \ \ldots\ldots, \ x_n = \frac{\det A_n}{\det A}$$

Solve:
$$\begin{cases} x + 2y - z = 1 \\ 3x + y + z = 2 \\ x + 0y - 5z = 0 \end{cases}$$

1) At the beginning we have to „tranform" the system of equation into a matrix and we have:

$$(*) \quad \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & 0 & -5 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

2) Then, we have to calculate the determinant of matrix A (we use Sarrus' method)

$$\det A : \left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 1 & 0 & -5 & 1 & 0 \end{array}\right] = (-5+2)-(-1+30) = -3+31 = 28$$

3) After that, we have to calculate $\det A_x$, $\det A_y$ and $\det A_z$, to do that we have to put the right side of the equation $(*)$ instead of x-column, y-column or z-column, and we have:

$$\det A_x = \left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 0 & -5 & 0 & 0 \end{array}\right] = (-5+0-0)-(-20) = -5+20 = 15 \Rightarrow \det A_x = 15$$

$$\det A_y = \left[\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & 1 \\ 3 & 2 & 1 & 3 & 2 \\ 1 & 0 & -5 & 1 & 0 \end{array}\right] = (-10+1+0)-(-2-15) = -9+17 = 8 \Rightarrow \det A_y = 8$$

$$\det A_z = \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 3 & 1 & 2 & 3 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] = (0+4+0)-(1+0+0) = 4-1 = 3 \Rightarrow \det A_z = 3$$

4) To compute x, y and z we have to put the results into the formulla and we have:

X: $x = \frac{\det A_x}{\det A} = \frac{15}{28}$

Y: $y = \frac{\det A_y}{\det A} = \frac{8}{28}$

Z: $z = \frac{\det A_z}{\det A} = \frac{3}{28}$

⇒ **ANSWER:**
$$\begin{cases} x = \frac{15}{28} \\ y = \frac{8}{28} \\ z = \frac{3}{28} \end{cases}$$

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