Structural Stability

Conversion from Imperfection-Sensitive into Imperfection-Insensitive Elastic Structures

Part I: Theory

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Motivation

Koiter’s initial postbuckling analysis in the context of the FEM

Symmetric bifurcation

Triples of values $\lambda_4, a_1$, and $a_1^*$

Consistently linearized eigenvalue problem

General case: nonlinear prebuckling paths

Special case: linear prebuckling paths

Completeness of solutions from Koiter’s initial postbuckling analysis

Conclusions
Motivation

- it is desirable to modify an imperfection-sensitive structure such that it becomes imperfection-insensitive

\[ S \approx S \]

\[ \lambda \]

\[ u \]

\[ \times \] example for an unsuitable strategy to achieve this goal.
Motivation

it is desirable to modify an imperfection-sensitive structure such that it becomes imperfection-insensitive

example for an suitable strategy to achieve this goal.
Koiter’s initial postbuckling analysis

Development of the function $G|_D$ as a Taylor series at the bifurcation point $C(u_C, \lambda_C)$:

$G^+(v, \eta) := G(\tilde{u}(\tilde{\lambda}(\eta)) + v, \tilde{\lambda}(\eta))$

$\Delta \lambda = \tilde{\lambda}(\eta) - \lambda_C$
\[
G^+ (v, \tilde{\lambda}) = \text{primary path} + \\
+ K_T \cdot v^+ + K_T,\lambda \cdot v^+ \Delta \lambda + \frac{1}{2} \tilde{K}_T,\lambda\lambda \cdot v^+ \Delta \lambda^2 + \cdots \\
+ \frac{1}{6} \tilde{K}_T,\lambda\lambda\lambda \cdot v^+ \Delta \lambda^3 + \frac{1}{24} K_T,\lambda\lambda\lambda\lambda \cdot v^+ \Delta \lambda^4 + \cdots \\
+ \frac{1}{2} K_T, v : v^+ \otimes v^+ + \frac{1}{2} K_T, u : v^+ \otimes v^+ \Delta \lambda + \cdots \\
+ \frac{1}{4} K_T, u\lambda\lambda : v^+ \otimes v^+ \Delta \lambda^2 + \frac{1}{12} K_T, u\lambda\lambda\lambda : v^+ \otimes v^+ \Delta \lambda^3 + \cdots \\
= 0
\]

- **Asymptotic developments** at the bifurcation point \( C(u_c, \lambda_c) \):

\[
v^+ (\eta) = v_1 \eta + v_2 \eta^2 + v_3 \eta^3 + v_4 \eta^4 + \cdots \\
\Delta \lambda (\eta) = \lambda_1 \eta + \lambda_2 \eta^2 + \lambda_3 \eta^3 + \lambda_4 \eta^4 + \cdots 
\]
Koiter’s initial postbuckling analysis

Notes:

- The following abbreviations are used:
  
  \[ K_{T, \text{uu}} : v_1 \otimes v_2 \rightarrow K_{T, \text{uu}} v_1 v_2 \]
  
  \[ K_{T, \text{uuu}} : v_1 \otimes v_2 \otimes v_3 \rightarrow K_{T, \text{uuu}} v_1 v_2 v_3 \]
  
  \[ K_{T, \text{u}} v = \frac{\partial K_T}{\partial u} v \]
  
  \[ \tilde{K}_{T, \lambda} = \frac{\partial K_T}{\partial u} \frac{d u}{d \lambda} \]
  
  \[ K_{T, \text{u} \lambda} v = \frac{\partial^2 K_T}{\partial u^2} \frac{d u}{d \lambda} v \]
  
  \[ \tilde{K}_{T, \lambda \lambda} = \frac{\partial^2 K_T}{\partial u^2} \frac{d u}{d \lambda} \frac{d u}{d \lambda} + \frac{\partial K_T}{\partial u} \frac{d^2 u}{d \lambda^2} \]

- The matrices in red color are zero in case of a linear prebuckling path!
Koiter’s initial postbuckling analysis

Notes:

- **calculation of the above matrices:**

\[
D_u K_T \cdot \xi = \frac{d}{d\alpha} K_T (u + \alpha \xi) \bigg|_{\alpha=0}
\]

\[
K_T : u \quad v = D_u K_T \cdot v = \frac{d}{d\alpha} K_T (u + \alpha v) \bigg|_{\alpha=0}
\]

\[
\tilde{K}_T : \lambda = D_u K_T \cdot \frac{du}{d\lambda} = \frac{d}{d\alpha} K_T \left(u + \alpha \frac{du}{d\lambda}\right) \bigg|_{\alpha=0}
\]

\[
K_T : u \lambda \cdot v = D_u \left[ D_u K_T \cdot \frac{du}{d\lambda}\right] \cdot v = \frac{d^2}{d\alpha d\beta} K_T \left(u + \alpha \frac{du}{d\lambda} + \beta v\right) \bigg|_{\alpha=0, \beta=0}
\]

\[
\tilde{K}_T : \lambda \lambda = K_T : uu \frac{du}{d\lambda} \frac{du}{d\lambda} + K_T : u \frac{d^2 u}{d\lambda^2} =
\]

\[
= \frac{d^2}{d\alpha^2} K_T \left(u + \alpha \frac{du}{d\lambda}\right) \bigg|_{\alpha=0} + \frac{d}{d\alpha} K_T \left(u + \alpha \frac{d^2 u}{d\lambda^2}\right) \bigg|_{\alpha=0}
\]
H. A. Mang

Koiter’s initial postbuckling analysis

\[ \begin{align*}
\eta \cdot (K_T v_1) + & \\
\eta^2 \cdot (K_T v_2 + \tilde{K}_T,\lambda v_1 \lambda_1 + \frac{1}{2} K_T, u v_1 v_1) + & \\
\eta^3 \cdot (K_T v_3 + \tilde{K}_T,\lambda v_1 \lambda_2 + K_T,\lambda v_2 \lambda_1 + \frac{1}{2} \tilde{K}_T,\lambda \lambda v_1 \lambda_1^2 + & \\
+ K_T, u v_1 v_2 + \frac{1}{2} K_T, u \lambda v_1 v_1 \lambda_1 + \frac{1}{6} K_T, u u v_1 v_1 v_1) + & \\
\eta^4 \cdot (K_T v_4 + \tilde{K}_T,\lambda v_1 \lambda_3 + \tilde{K}_T,\lambda v_2 \lambda_2 + \tilde{K}_T,\lambda v_3 \lambda_1 + \tilde{K}_T,\lambda \lambda v_1 \lambda_1 \lambda_2 + & \\
+ \frac{1}{2} \tilde{K}_T,\lambda \lambda v_2 \lambda_1^2 + \frac{1}{6} \tilde{K}_T,\lambda \lambda \lambda v_1 \lambda_1^3 + K_T, u v_1 v_3 + \frac{1}{2} K_T, u v_2 v_2 + & \\
+ \frac{1}{2} K_T, u \lambda v_1 v_1 \lambda_2 + K_T, u \lambda v_1 v_2 \lambda_1 + \frac{1}{4} K_T, u \lambda \lambda v_1 v_1 \lambda_1^2 + & \\
+ \frac{1}{2} K_T, u u v_1 v_1 v_2 + \frac{1}{6} K_T, u u \lambda v_1 v_1 v_1 \lambda_1 + \frac{1}{24} K_T, u u u v_1 v_1 v_1 v_1) + & \\
\vdots \quad O(\eta^5) \quad & = 0
\end{align*} \]
Koiter’s initial postbuckling analysis

Calculation of the coefficients $\lambda_i$, $i = 1, 2, \ldots, 5$

- from the coefficient of $\eta$: $K_T v_1 = 0 \Rightarrow v_1 \ldots$ eigenvector
- from the coefficient of $\eta^2$ \[ \Rightarrow \lambda_1 \rightarrow v_2 \]
  Premultiplication with $v_1^T$ yields \[ \lambda_1 = b_0 = -\frac{1}{2} \frac{v_1^T K_T, u v_1 v_1}{v_1^T \tilde{K}_T, \lambda v_1} \]
- from the coefficient of $\eta^3$ \[ \Rightarrow \lambda_2 \rightarrow v_3 \]
  Premultiplication with $v_1^T$ yields \[ \lambda_2 = a_1 \lambda_1^2 + b_1 \lambda_1 + d_1 \quad \text{with} \quad a_1 = -\frac{1}{2} \frac{v_1^T \tilde{K}_T, \lambda \lambda v_1}{v_1^T \tilde{K}_T, \lambda v_1} \] (1)

$a_1 \ldots "\text{nonlinearity coefficient}"

trivially zero in case of linear prebuckling path \[ \Rightarrow \tilde{K}_T, \lambda \lambda = 0 \]
from the coefficient of $\eta^4 \Rightarrow \lambda_3 \rightarrow \mathbf{v}_4$

Premultiplication with $\mathbf{v}_1^T$ yields

$$\lambda_3 = a_1^* \lambda_1^3 + b_1^* \lambda_1^2 + c_1^* \lambda_1 + e_1^*$$

$$a_1^* = -\frac{1}{6} \frac{\mathbf{v}_1^T \mathbf{K}_T ; \lambda \lambda \lambda \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{\tilde{K}}_T , \lambda \mathbf{v}_1}$$

from the coefficient of $\eta^5 \Rightarrow \lambda_4 \rightarrow \mathbf{v}_5$

Premultiplication with $\mathbf{v}_1^T$ yields

$$\lambda_4 = \hat{a}_1 \lambda_1^4 + \hat{b}_1 \lambda_1^3 + \hat{c}_1 \lambda_1^2 + \hat{d}_1 \lambda_1 + \hat{f}_1$$

$$\hat{a}_1 = -\frac{1}{24} \frac{\mathbf{v}_1^T \mathbf{K}_T ; \lambda \lambda \lambda \lambda \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{\tilde{K}}_T , \lambda \mathbf{v}_1}$$

from the coefficient of $\eta^6 \Rightarrow \lambda_5 \rightarrow \mathbf{v}_6$

Premultiplication with $\mathbf{v}_1^T$ yields

$$\lambda_5 = \tilde{a}_1 \lambda_1^5 + \tilde{b}_1 \lambda_1^4 + \tilde{c}_1 \lambda_1^3 + \tilde{d}_1 \lambda_1^2 + \tilde{e}_1 \lambda_1 + \tilde{g}_1$$

$$\tilde{a}_1 = -\frac{1}{120} \frac{\mathbf{v}_1^T \mathbf{K}_T ; \lambda \lambda \lambda \lambda \lambda \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{\tilde{K}}_T , \lambda \mathbf{v}_1}$$
Symmetric bifurcation \( \lambda_1 = \lambda_3 = \lambda_5 = \cdots = 0 \)

The equations (1.1), (2), (3), and (4) can be rewritten as:

\[
a_1 \lambda_1^2 + b_1 \lambda_1 + c_1 = 0 \quad \text{with} \quad c_1 = d_1 - \lambda_2
\]
\[
a_1^* \lambda_1^3 + b_1^* \lambda_1^2 + c_1^* \lambda_1 + d_1^* = 0 \quad \text{with} \quad d_1^* = e_1^* - \lambda_3
\]
\[
\hat{a}_1 \lambda_1^4 + \hat{b}_1 \lambda_1^3 + \hat{c}_1 \lambda_1^2 + \hat{d}_1 \lambda_1 + \hat{e}_1 = 0 \quad \text{with} \quad \hat{e}_1 = \hat{f}_1 - \lambda_4
\]
\[
\tilde{a}_1 \lambda_1^5 + \tilde{b}_1 \lambda_1^4 + \tilde{c}_1 \lambda_1^3 + \tilde{d}_1 \lambda_1^2 + \tilde{e}_1 \lambda_1 + \tilde{f}_1 = 0 \quad \text{with} \quad \tilde{f}_1 = \tilde{g}_1 - \lambda_5
\]

◊ the underlined coefficients must vanish (for symmetric bifurcation)

from \( \tilde{e}_1 = 0 \) \( \Rightarrow \)

\[
a_1 \lambda_2^2 + b_2 \lambda_2 + d_3 - \lambda_4 = 0
\] (5)

◊ some of the remaining coefficients may vanish

if \( c_1^* = c_1^*(\kappa) = 0 \) \( \Rightarrow \) (5) disintegrates into

\[
2a_1 \lambda_2 + b_2 = 0 \quad a_1 \lambda_2^2 - d_3 + \lambda_4 = 0
\] (6)
Symmetric bifurcation

- symmetric bifurcation from nonlinear prebuckling paths is associated either with

relation I

\[ v_j^T \tilde{K}_T,\lambda \lambda v_1 = 0, \quad j \neq 1 \]

or with

relation II

\[ v_1^T \tilde{K}_T,\lambda \lambda \lambda v_1 = 0 \quad \rightarrow \quad a_1^* = 0 \]

\( v_j^* \ldots j\text{-th eigenvalue of the so-called } \text{consistently linearized eigenproblem} \)

- Relation I occurs together either with \( c_1^* \neq 0 \) (5) or \( c_1^* = 0 \) (6)
- Relation II only occurs together with \( c_1^* = 0 \) (6)
Symmetric bifurcation

- For relation I: \( v_j^T \tilde{K}_T,\lambda \lambda v_1 = 0, \ j \neq 1 \) \( \lambda_2 = 0, \ k = 1, 2, \ldots \ \Rightarrow \tilde{\lambda}(\eta) = \lambda_C = \text{const.} \)

- For relation II: \( v_1^T \tilde{K}_T,\lambda \lambda v_1 = 0 \) \( \Rightarrow a_1 = 0, \lambda_2 = 0, \lambda_4 < 0 \)

- For relation I and relation II: \( \tilde{K}_T,\lambda \lambda v_1 = 0 \) \( \Rightarrow a_1 = 0, \lambda_2 = 0, \lambda_4 = 0, \lambda_6 < 0 \)
Triples of values $\lambda_4$, $a_1$, and $a_1^*$

- the following triples of values $\lambda_4$, $a_1$, $a_1^*$ are obtained:
  
  $\lambda_4 = 0$, $a_1 < 0$, $a_1^* < 0$, $a_1^* > 0$,
  
  $\lambda_4 = 0$, $a_1 = 0$ (with $\bar{K}_{T,\lambda\lambda}v_1 = 0$), $a_1^* = 0$, $a_1^* > 0$,
  
  $\lambda_4 < 0$, $a_1 = 0$ (with $\bar{K}_{T,\lambda\lambda}v_1 \neq 0$), $a_1^* = 0$, $a_1^* > 0$,
  
  $\lambda_4 = 0$, $a_1 = 0$ (with $\bar{K}_{T,\lambda\lambda}v_1 
eq 0$), $a_1^* = 0$, $a_1^* > 0$.

\[ v_j^* T K_{T,\lambda\lambda} v_1 = 0, j \neq 1 \]
\[ a_1^* < 0 \]
Triples of values $\lambda_4$, $a_1$, and $a_1^*$

- five octants as geometric loci of triples of values $(\lambda_2, \lambda_4, a_1)$:
  - octants I, II, and IV:
    $$v_1^T \tilde{K}_T, \lambda v_1 = -1 \quad \text{and} \quad v_1^T \tilde{K}_T, \lambda \lambda v_1 \geq 0$$
  - octants V and VII:
    $$v_1^T \tilde{K}_T, \lambda v_1 = -1 \quad \text{and} \quad v_1^T \tilde{K}_T, \lambda \lambda v_1 \leq 0$$
  - octant VII, alternatively:
    $$v_1^T \tilde{K}_T, \lambda v_1 = +1 \quad \text{and} \quad v_1^T \tilde{K}_T, \lambda \lambda v_1 \geq 0$$
Consistently linearized Eigenproblem

- **Aim**: specific geometric properties of the eigenvalue curve of this problem for different situations at point $T \rightarrow \lambda_2 = 0$

- Investigation of the coefficient $a_1$

  - condition for the stability limit $\lambda = \lambda_S$ on nonlinear load-displacement paths:

    $$K_T(u(\lambda_S)) \mathbf{v} = 0$$

- Taylor series expansion of $K_T(\lambda_S)$ at $\lambda < \lambda_S$:

  $$\left[ K_T + (\lambda^* - \lambda) + \frac{1}{2} (\lambda^* - \lambda)^2 \tilde{K}_{T,\lambda\lambda} + \cdots \right] \mathbf{v} = 0$$
Motivation
Koiter
Symmetric bifurcation

Triples of values $\lambda_4, a_1, \text{ and } a_1^-$

Consistently linearized eigenvalue problem

General case: nonlinear prebuckling paths

Special case: linear prebuckling paths

Completeness of solutions from Koiter's initial postbuckling analysis

Conclusions

Consistently linearized Eigenproblem

- neglecting terms of higher than first order in $(\lambda_S - \lambda)$ yields the so-called

$$\begin{bmatrix} K_T + (\lambda^* - \lambda) \tilde{K}_{T,\lambda} \end{bmatrix} v^* = 0$$

$\lambda^*$: estimate of $\lambda_S$

$v^*$: eigenvector


- because of the singularity of $K_T$ at the stability limit,

$$\lambda^* - \lambda = 0 \quad \rightarrow \quad \lambda^* = \lambda_1^*, \quad \lambda = \lambda_S, \quad v^* = v_1^* = v_1$$

- derivation of (8) with respect to $\lambda$ yields

$$\left[ \lambda^*, \tilde{K}_T, \lambda + (\lambda^* - \lambda) \tilde{K}_T, \lambda, \lambda \right] v^* + \left[ K_T + (\lambda^* - \lambda) \tilde{K}_T, \lambda \right] v^*, \lambda = 0$$
At a **stability limit** in form of a bifurcation point, in general,

\[ \lambda^*_{,\lambda} = 0, \quad v^*_{1,\lambda} = a_1 v_1 \quad (10) \]

- Derivation of (9) with respect to \( \lambda \), specialization for the stability limit and premultiplication by \( v^T_1 \) gives

\[ \lambda^*_{,\lambda\lambda} v^T_1 \tilde{K}_{T,\lambda} v_1 - v^T_1 \tilde{K}_{T,\lambda\lambda} v_1 = 0 \quad (11) \]

Comparing (11) with (1.2) shows that

\[ \lambda^*_{,\lambda\lambda} = 2 a_1 \quad (12) \]

- Specialization of (10.2) and (12) for \( a_1 = 0 \) yields

\[ v^*_{1,\lambda} = 0, \quad \lambda^*_{1,\lambda\lambda} = 0 \quad (13) \]
**Motivation**

Koiter

Symmetric bifurcation

Triples of values $\lambda_4, a_1$, and $a_1$

Consistently linearized eigenvalue problem

**General case:** nonlinear prebuckling paths

**Special case:** linear prebuckling paths

Completeness of solutions from Koiter's initial postbuckling analysis

**Conclusions**

$\lambda_2 - \lambda_4 - a_1$ curves containing point $T \rightarrow \lambda_2 = 0$

---

**Mode a:**

Horizontal projection

**Mode b:**

Horizontal projection
\(\lambda_2 - \lambda_4 - a_1\) curves containing point \(T \rightarrow \lambda_2 = 0\)

**Mode c:**

\[ \begin{align*}
F^0 & \rightarrow H^0 \\
S^0 & \rightarrow S^0
\end{align*} \]

Horizontal projection

**Mode d:**

\[ \begin{align*}
F & \rightarrow F^0 \\
S & \rightarrow S^0 \\
T & \rightarrow T
\end{align*} \]

Horizontal projection

\(H \ldots\) coincidence of bifurcation point and snap-through point
$\lambda_2 - \lambda_4 - a_1$ curves containing point $T \rightarrow \lambda_2 = 0$

Mode $e$:

Mode $f$:
\( \lambda_2 - \lambda_4 - a_1 \) curves containing point \( T \rightarrow \lambda_2 = 0 \)

**Mode \( g \):**

- \( S = F = N = T \)
- \( \lambda_2 \) and \( \lambda_4 \)

**Mode \( h \):**

- \( S = T \)
- \( F = N = T \)
- \( \lambda_2 \) and \( \lambda_4 \)
another distinctive feature of these curves follows from the first two partial derivatives of

$$2a_1 \lambda_2 + b_2 = 0$$

with respect to a design parameter $\kappa$:

$$2 (a_1, \kappa \lambda_2 + a_1 \lambda_2, \kappa) + b_2, \kappa = 0$$

and

$$2 (a_1, \kappa \kappa \lambda_2 + 2a_1, \kappa \lambda_2, \kappa + a_1 \lambda_2, \kappa \kappa) + b_2, \kappa \kappa = 0$$

Specialization for $\lambda_2 = 0$ and $a_1 = 0$ gives

$$b_2, \kappa = 0 \quad \text{and} \quad 2a_1, \kappa \lambda_2, \kappa + b_2, \kappa \kappa = 0$$
### Sensitivity analysis

**Values of** $\lambda_2, \kappa$, $\lambda_4$, and $a_1$ **for points** $T(\lambda_2 = 0, \lambda_4, a_1)$

<table>
<thead>
<tr>
<th>Mode ( )</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2, \kappa$</td>
<td>$\neq 0$</td>
<td>$\neq 0$</td>
<td>0</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
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<td>0</td>
<td>0</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$&lt; 0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode ( )</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2, \kappa$</td>
<td>$\lambda_2(\kappa) \geq 0$</td>
<td>$\lambda_2(\kappa) = 0$</td>
<td>$\lambda_2(\kappa) = 0$</td>
<td>$\lambda_2(\kappa) = 0$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>$\lambda_4(\kappa) = 0$</td>
<td>$\lambda_4(\kappa) = 0$</td>
<td>$\lambda_4(\kappa) = 0$</td>
<td>$\lambda_4(\kappa) &lt; 0$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$a_1(\kappa) \geq 0$</td>
<td>$a_1(\kappa) &lt; 0$</td>
<td>$a_1(\kappa) = 0$</td>
<td>$a_1(\kappa) = 0$</td>
</tr>
</tbody>
</table>
Discussion of Modes (a) – (h)

Mode (a)

- at point $T$
  \[ \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots \]
  \[ a_1 < 0, \quad a_1^* < 0 \]
  \[ v_1^*; \lambda \lambda = 3 (a_1^2 + a_1^*) v_1 \]

- at point $I$
  \[ \lambda_2 > 0, \quad \lambda_4 > 0, \quad \lambda_6 > 0, \ldots \]
  \[ \ddot{K}_T; \lambda \lambda v_1 = 0 \quad \rightarrow \quad a_1 = 0, \]
  \[ a_1^* > 0 \]

✔ conversion from imperfection sensitivity into insensitivity
Mode (b) at point $T$

\[ \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots \]

\[ \tilde{K}_T;\lambda\lambda \mathbf{v}_1 = 0 \quad \rightarrow \quad a_1 = 0, \quad a_1^* > 0 \]

\[ \mathbf{v}_1^*;\lambda\lambda = 3a_1^*\mathbf{v}_1 \]

Conversion from imperfection sensitivity into insensitivity
Discussion of Modes (a) – (h)

○ Mode (c)

- at point $T$
  \[ \tilde{K}_{T;\lambda\lambda} v_1 = 0 \quad \rightarrow \quad a_1 = 0 \]
  \[ v_1^*,\lambda = 0, \quad v_1^*,\lambda\lambda = 0 \]
  \[ v_1^*,\lambda\lambda = 0 \]

- at point $H$ (hilltop bifurcation)
  \[ v_1^*,\lambda = 0, \quad v_1^*,\lambda\lambda = 0, \]
  \[ \lambda_2 < 0, \quad \lambda_4 < 0, \]
  \[ \lambda_1^*,\lambda = -1, \quad a_1 = -\infty \]

-x no conversion from imperfection sensitivity into insensitivity
  \[ \iff \quad \text{transition from } \lambda_4 > 0 \text{ to } \lambda_4 < 0 \]
Discussion of Modes (a) – (h)

**Mode (d)**

At point $T$

$$\lambda_2 = 0, \quad \lambda_4 < 0,$$

$$v_1^T \tilde{K}_T;\lambda \lambda v_1 = 0 \quad \rightarrow \quad a_1 = 0$$

$$v_1^T \tilde{K}_T;\lambda \lambda v_1 = 0 \quad \rightarrow \quad a_1^* = 0$$

$$v_1^*, \lambda \lambda = \sum_{j=2}^{n} c_{1j} \lambda v_j^*$$

- conversion from imperfection sensitivity into insensitivity
Discussion of Modes (a) – (h)

Mode (e)

at point $T$

$\lambda_2 = 0, \quad \lambda_4 = 0,$

$a_1 = 0$

$\nu^*_1, \lambda = 0$

conversion from imperfection sensitivity into insensitivity
Discussion of Modes (a) – (h)

**Mode (f)**

- at point $F = N = T$
  - $\lambda, \xi = 0$, $\lambda, \xi \xi = 0$,
  - $v_1 = 0$,
  - $\lambda, \lambda = -1$, $a_1 = -\infty$
  - $v_1^*, \lambda \lambda = 3 (a_1^2 + a_1^*) v_1$
  - $\tilde{K}_T d\tilde{u} = 0$

✅ transition from bifurcation buckling to no buckling
Discussion of Modes (a) – (h)

**Mode (g) (von Mises truss)**

- at point $F = N = T$
  $$\lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots$$
  $$\tilde{K}_T \lambda \lambda v_1 = 0 \quad \rightarrow \quad a_1 = 0,$$
  $$\tilde{K}_T d\tilde{u} = 0$$
  $$v_1^*, \lambda \lambda = 3 a_1^* v_1$$

- "final situation": saddle point of higher order
  $$\lambda_1^* - \lambda = 0, \quad \lambda_1^*, \lambda = 0, \quad \lambda_1^*, \lambda \lambda = 0$$
  $$\lambda_1^*, \lambda \lambda \lambda = 0, \quad \lambda_1^*, \lambda \lambda \lambda \lambda = 0$$

- transition from bifurcation buckling to no buckling
Discussion of Modes (a) – (h)

**Mode (g) (cylindrical panel)**

- at point \( F = N = T \)
  \[
  \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots
  \]
  \[
  \tilde{K}_T;_{\lambda\lambda} v_1 = 0 \quad \rightarrow \quad a_1 = 0,
  \]
  \[
  \tilde{K}_T d\tilde{u} = 0
  \]
  \[
  v_1^*,_{\lambda\lambda} = 0
  \]

- “final situation”: saddle point of higher order
  \[
  \lambda^*_1 - \lambda = 0, \quad \lambda^*_1;_{\lambda\lambda} = 0, \quad \lambda^*_1;_{\lambda\lambda\lambda\lambda} = 0
  \]
  \[
  \lambda^*_1;_{\lambda\lambda\lambda\lambda} = 0, \quad \lambda^*_1;_{\lambda\lambda\lambda\lambda\lambda\lambda} = 0
  \]

- transition from bifurcation buckling to no buckling
Mode (h)

at point $T$

\[ \lambda_2 = 0, \quad \lambda_4 < 0, \]

\[ a_1(\kappa) = 0, \]

\[ \mathbf{v}_1^T \tilde{\mathbf{K}}_{T,\lambda\lambda} \mathbf{v}_1 = 0 \quad \rightarrow \quad a_1 = 0, \]

\[ \mathbf{v}_1^T \tilde{\mathbf{K}}_{T,\lambda\lambda\lambda} \mathbf{v}_1 = 0 \quad \rightarrow \quad a_1^* = 0 \]

\[ \mathbf{v}_1^*,\lambda\lambda = \sum_{j=2}^{n} c_{1j,\lambda} \mathbf{v}_j^* \]

transition from bifurcation buckling to no buckling
Linear prebuckling paths

For the special case of buckling from linear prebuckling paths
\[ \tilde{K}_{T,\lambda\lambda} = 0 \rightarrow a_1 = 0, \quad \tilde{K}_{T,\lambda\lambda\lambda} = 0 \rightarrow a_1^* = 0, \quad \ldots \]

Therefore, relation I
\[ v_j^T \tilde{K}_{T,\lambda\lambda} v_1 = 0, \quad j \neq 1, \]
and relation II
\[ v_1^T \tilde{K}_{T,\lambda\lambda\lambda} v_1 = 0 \rightarrow a_1^* = 0 \]
are satisfied trivially.

Therefore, there are no restrictions on the plane curves \( \lambda_2 = \lambda_2(\kappa), \lambda_4 = \lambda_4(\kappa) \), analogous to the ones for the general case!
Completeness of solutions

For bifurcation from nonlinear prebuckling paths

\[ \mathbf{v}_{1}^{*}, \lambda = a_{1} \mathbf{v}_{1} \]

\[ \mathbf{v}_{1}^{*}, \lambda \lambda = 3 \left( a_{1}^{2} + a_{1}^{*} \right) \mathbf{v}_{1} + \sum_{j=2}^{n} c_{1j}, \lambda \mathbf{v}_{j}^{*} \]

Following from relation I and relation II, respectively, \( \mathbf{v}_{1}^{*}, \lambda \lambda \) disintegrates into

\[ \mathbf{v}_{1}^{*}, \lambda \lambda = 3 \left( a_{1}^{2} + a_{1}^{*} \right) \mathbf{v}_{1}, \quad c_{1j}, \lambda = 0, \quad j \neq 0 \] (14)

or

\[ \mathbf{v}_{1}^{*}, \lambda \lambda = 3 a_{1}^{2} \mathbf{v}_{1} + \sum_{j=2}^{n} c_{1j}, \lambda \mathbf{v}_{j}^{*} \] (15)
Completeness of solutions

Mode (a): \(v_1^*, \lambda = a_1 v_1\), \(v_1^*, \lambda \lambda = 3 \left( a_1^2 + a_1^* \right) v_1\)

Mode (b): \(v_1^*, \lambda = 0\), \(v_1^*, \lambda \lambda = 3 a_1^* v_1\)

Mode (c): \(v_1^*, \lambda = 0\), \(v_1^*, \lambda \lambda = 0\)

Mode (d): \(v_1^*, \lambda = 0\), \(v_1^*, \lambda \lambda = \sum_{j=2}^{n} c_{1j} \lambda v_j^*\)

Mode (e): same as Mode (c)

Mode (f): same as Mode (a)

Mode (g): same as Mode (b) or Mode (c)

Mode (h): same as Mode (d)

\(\Rightarrow\) Mode (a) and Mode (b) represent a complete subset of (I)

\(\Rightarrow\) Mode (c) and Mode (d) represent a complete subset of (II)
Conclusions

- Conversion from imperfection-sensitive into imperfection-insensitive structures requires symmetric bifurcation.

- Symmetric bifurcation from nonlinear prebuckling paths is associated either with $v_j^T \tilde{K}_T,\lambda \lambda v_1 = 0$, $j \neq 1$ or with $v_1^T \tilde{K}_T,\lambda \lambda \lambda v_1 = 0$.

- The geometric loci of all points in the $\lambda_2, \lambda_4, a_1$ space are solutions of $\lambda_4 = a_1 \lambda_2^2 + b_2 \lambda_2 + d_3$. For $\lambda_2 = 0$, they are restricted to the two half-axes $\lambda_4 \leq 0$ and $a_1 \leq 0$.

- $\lambda_2 = 0$ is a necessary but not sufficient condition for the transition to imperfection insensitivity.
Structural Stability

Conversion from Imperfection-Sensitive into Imperfection-Insensitive Elastic Structures
Part II: Numerical Analyses

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Example 1: Pin-jointed bar with two degrees of freedom

Example 2: von Mises truss

Example 3: cylindrical panel

Example 4: Pin-jointed bar with linear prebuckling paths

Conclusions
Pin-jointed bar with two degrees of freedom

design parameters:
- spring stiffness
- initial rise
Pin-jointed bar – increase of spring stiffness

conversion from imperfection sensitivity into insensitivity

c_{1} = 0

c_{1} = 1.5

c_{1} \approx 5.6

load-displacement paths

\lambda^{*} - \lambda \text{ curves}
Pin-jointed bar – increase of spring stiffness

\( \lambda_2 - \lambda_4 - a_1 \) curve

Mathematical conditions

- At point \( T \)
  \[ \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots \]
  \[ a_1 < 0, \quad a_1^* < 0, \]
  \[ v_1^*, \lambda \lambda = 3 \left( a_1^2 + a_1^* \right) v_1 \]

- At point \( I \)
  \[ \lambda_2 > 0, \quad \lambda_4 > 0, \quad \lambda_6 > 0, \ldots \]
  \[ \tilde{K}_{T, \lambda \lambda} v_1 = 0 \quad \Rightarrow \quad a_1 = 0, \]
  \[ a_1^* > 0 \]
Pin-jointed bar – decrease of initial rise

- transition from bifurcation buckling to no buckling

\[ \phi_0 = 0.90 \quad \phi_0 = 0.70 \quad \phi_0 \approx 0.55 \quad \phi_0 = 0.50 \]

load-displacement paths

\[ \lambda^* - \lambda \text{ curves} \]
Pin-jointed bar – decrease of initial rise

\[ \lambda_2 - \lambda_4 - a_1 \] curve

\[ \mathbf{v}_1^*, \lambda \lambda = 3 \left( a_1^2 + a_1^* \right) \mathbf{v}_1 \]

- at point \( F = N = T \)

\[ \lambda, \xi = 0, \quad \lambda, \xi \xi = 0, \]
\[ \mathbf{v}_1 = 0, \]
\[ \lambda, \lambda = -1, \quad a_1 = -\infty, \]
\[ \tilde{K}_T d\tilde{u} = 0 \]
von Mises truss

design parameters:

- spring stiffness
- initial rise

load displacement path:

$\lambda$-$\lambda$ curve:
von Mises truss – increase of spring stiffness

conversion from imperfection sensitivity into insensitivity

\[ c = 24 \text{ kN/cm} \]
\[ c \approx 40.8 \text{ kN/cm} \]
\[ c = 100 \text{ kN/cm} \]

load-displacement paths

\[ \lambda^* - \lambda \] curves
von Mises truss – increase of spring stiffness

\[ \lambda_2 - \lambda_4 - a_1 \] curve

mathematical conditions

- at point \( T \)
  \[ \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \quad \ldots \]
  \[ \tilde{K}_{T, \lambda \lambda} v_1 = 0 \quad \rightarrow \quad a_1 = 0, \quad a_1^* > 0, \]
  \[ v_1^*, \lambda \lambda = 3a_1^* v_1 \]
von Mises truss – decrease of initial rise

borderline case

transition from bifurcation buckling to no buckling

$h = 40$ cm

$h = 32$ cm

$h \approx 25.2$ cm

load-displacement paths

$\lambda^*-\lambda$ curves
von Mises truss – decrease of initial rise

\( \lambda_2 - \lambda_4 - a_1 \) curve

mathematical conditions

- at point \( F = N = T \)
  \[ \lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots \]
  \[ \tilde{K}_{T,\lambda\lambda} v_1 = 0 \quad \rightarrow \quad a_1 = 0, \]
  \[ \tilde{K}_T d\tilde{u} = 0, \]
  \[ v_{1*,\lambda\lambda} = 3a_{1*}v_1 \]

- saddle point of \( \lambda^*(\lambda) \)
  \[ \lambda_1^* - \lambda = 0, \quad \lambda_1^*,\lambda = 0, \quad \lambda_1^*,\lambda\lambda = 0 \]

- “final situation”:
  saddle point of higher order
  \[ \lambda_1^* - \lambda = 0, \quad \lambda_1^*,\lambda = 0, \quad \lambda_1^*,\lambda\lambda = 0, \]
  \[ \lambda_1^*,\lambda\lambda\lambda = 0 \]
imperfection-insensitive truss

transition from bifurcation buckling to no buckling

Conclusions
H. A. Mang

pin-jointed bar
von Mises truss
cylindrical shell
linear prebuckling path
Conclusions

cylindrical panel

design parameters:
- thickness
- spring stiffness
- initial rise

load displacement path:

\[ \lambda = \lambda^* \]

\[ \lambda^* - \lambda \text{ curve:} \]
cylindrical panel – increase of shell thickness

\[ t = 3.35 \text{ cm} \]

\[ t \approx 6.35 \text{ cm} \]

\[ t = 7.35 \text{ cm} \]

\[ \lambda \]

\[ u \]

\[ \lambda^* \]

\[ \lambda^* = \lambda \]

\[ \lambda_1^* = \lambda \]

\[ \lambda^* = \lambda \]

\[ \lambda^* = \lambda \]

\[ C \]

\[ D \]

no conversion from imperfection sensitivity into insensitivity

**Pin-jointed bar**

von Mises truss

cylindrical shell

linear prebuckling path

Conclusions
cylindrical panel – increase of shell thickness

- no conversion from imperfection sensitivity into insensitivity

\[ t = 3.35 \text{ cm} \quad t \approx 6.35 \text{ cm} \quad t = 7.35 \text{ cm} \]

details of load-displacement paths

\[ \lambda^* = \lambda \]

details of \( \lambda^*-\lambda \) curves
cylindrical panel – increase of shell thickness

$\lambda_2 - \lambda_4 - a_1$ curve  

mathematical conditions

- at point $T$
  
  $\tilde{K}_{T,\lambda\lambda} v_1 = 0 \rightarrow a_1 = 0,$
  
  $v_1^{*},\lambda = 0,$  
  
  $v_1^{*},\lambda\lambda = 0,$
  
  $v_1^{*},\lambda\lambda = 0$
cylindrical panel – increase of spring stiffness

$t = 6.35\ \text{cm}$

✓ conversion from imperfection sensitivity into insensitivity

\[
c = 0 \\
\lambda = \lambda^* = \lambda \\
\]

\[
c = 75\ \text{kN/cm} \\
\lambda = \lambda^* \\
\]

\[
c = 200\ \text{kN/cm} \\
\lambda = \lambda^* \\
\]

load-displacement paths

details of $\lambda^* - \lambda$ curves
Conclusions

cylindrical panel – increase of spring stiffness

\( t = 6.35 \text{ cm} \)

✔ conversion from imperfection sensitivity into insensitivity

\[ c = 0 \quad \quad c = 75 \text{ kN/cm} \quad \quad c = 200 \text{ kN/cm} \]

details of load-displacement paths

details of \( \lambda^* - \lambda \) curves
cylindrical panel – increase of spring stiffness

\[ \lambda_2 - \lambda_4 - a_1 \] curve

mathematical conditions

- at point \( T \)
  \[ \lambda_2 = 0, \quad \lambda_4 = 0, \]
  \[ a_1 = 0, \]
  \[ v_1^* \lambda \lambda = 0 \]
cylindrical panel – increase of spring stiffness

\( t = 7.35 \text{ cm} \)

✓ conversion from imperfection sensitivity into insensitivity

\[
c = 0 \quad c \approx 15 \text{ kN/cm} \quad c = 50 \text{ kN/cm}
\]

\[
\begin{align*}
\lambda & \quad \lambda & \quad \lambda \\
5 \quad 10 \quad 25 \quad 30 & \quad 5 \quad 10 \quad 20 \quad 25 \quad 30 & \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30
\end{align*}
\]

load-displacement paths

details of \( \lambda^* - \lambda \) curves
cylindrical panel — increase of spring stiffness

\[ t = 7.35 \, \text{cm} \]

✔ conversion from imperfection sensitivity into insensitivity

\[
c = 0 \quad c \approx 15 \, \text{kN/cm} \quad c = 50 \, \text{kN/cm}
\]

load-displacement paths

details of \( \lambda^* - \lambda \) curves
cylindrical panel – increase of spring stiffness

\[ \lambda_2 - \lambda_4 - a_1 \] curve

mathematical conditions

- at point \( T \)
  \[ \lambda_2 = 0, \quad \lambda_4 < 0, \]
  \[ v^T_1 \tilde{K}_{T,\lambda\lambda} v_1 = 0 \rightarrow a_1 = 0, \]
  \[ v^T_1 \hat{K}_{T,\lambda\lambda\lambda} v_1 = 0 \rightarrow a_1^* = 0, \]
  \[ v^*_1,\lambda\lambda = \sum_{j=2}^{n} c_{1j,\lambda} v^*_j \]
cylindrical panel – decrease of initial rise

\[ t = 6.35 \text{ cm}, \quad c = 0 \]

transition from bifurcation buckling to no buckling

\[ h = 8 \text{ cm} \quad h = 6 \text{ cm} \quad h \approx 4.0 \text{ cm} \]

load-displacement paths

details of \( \lambda^* - \lambda \) curves
cylindrical panel – decrease of initial rise

\[ t = 6.35 \text{ cm}, \quad c = 0 \]

\[ \rightarrow \text{transition from bifurcation buckling to no buckling} \]

\[ h = 8 \text{ cm} \quad h = 6 \text{ cm} \quad h \approx 4.0 \text{ cm} \]

Details of load-displacement paths

Details of \( \lambda^* - \lambda \) curves
cylindrical panel – decrease of initial rise

\[ \lambda_2 - \lambda_4 - a_1 \text{ curve} \]

planar point of \( \lambda^*(\lambda) \)

\[
\lambda_1^* - \lambda = 0, \quad \lambda_1^*,\lambda = 0, \\
\lambda_1^*,\lambda\lambda = 0, \quad \lambda_1^*,\lambda\lambda\lambda = 0
\]

mathematical conditions

at point \( F = N = T \)

\[
\lambda_2 = 0, \quad \lambda_4 = 0, \quad \lambda_6 = 0, \ldots
\]

\[
\tilde{K}_{T,\lambda\lambda} v_1 = 0 \quad \rightarrow \quad a_1 = 0,
\]

\[
\tilde{K}_T d\ddot{u} = 0,
\]

\[
v_1^*,\lambda\lambda = 0
\]

“final situation”:

saddle point of higher order

\[
\lambda_1^* - \lambda = 0, \quad \lambda_1^*,\lambda = 0, \\
\lambda_1^*,\lambda\lambda = 0, \quad \lambda_1^*,\lambda\lambda\lambda = 0, \\
\lambda_1^*,\lambda\lambda\lambda\lambda = 0
\]
cylindrical panel – decrease of initial rise

$t = 7.35 \text{ cm}, \, c = 15$

transition from bifurcation buckling to no buckling

$h = 9 \text{ cm} \quad h = 7 \text{ cm} \quad h \approx 6.0 \text{ cm}$

load-displacement paths

details of $\lambda^*-\lambda$ curves
cylindrical panel — decrease of initial rise

\( t = 7.35 \text{ cm, } c = 15 \)

\( \Rightarrow \) transition from bifurcation buckling to no buckling

\( h = 9 \text{ cm} \)  \hspace{1cm} \( h = 7 \text{ cm} \)  \hspace{1cm} \( h \approx 6.0 \text{ cm} \)

details of load-displacement paths

\[ \lambda \]
\[ \lambda^* \]
\[ \lambda_1^* \]

details of \( \lambda^*-\lambda \) curves

\( \lambda^* = \lambda \)
cylindrical panel – decrease of initial rise

\[ \lambda_2 - \lambda_4 - a_1 \] curve

Mathematical conditions

- at point \( T \):
  \[ \lambda_2 = 0, \quad \lambda_4 < 0, \]
  \[ a_1(\kappa) = 0, \]
  \[ v_1^T \tilde{K}_{T,\lambda \lambda} v_1 = 0 \quad \Rightarrow \quad a_1 = 0, \]
  \[ v_1^T \tilde{K}_{T,\lambda \lambda \lambda} v_1 = 0 \quad \Rightarrow \quad a_1^* = 0, \]

\[ v_1^*,\lambda \lambda = \sum_{j=2}^{n} c_{1,j,\lambda} v_j^* \]
Pin-jointed bar with linear prebuckling paths

design parameters:
- spring stiffness
- length of one bar
Pin-jointed bar with linear prebuckling paths

**load coefficients:**

\[ \lambda_2 = \frac{1}{6k(1+k)l^3} \left[ (-3 + 3k + k^2)c_\phi - 3l^2(1 - k + k^2)c_2 + 3k^2l^2c_1 \right], \]

\[ \lambda_4 = \frac{1}{120k^3(1+k)l^5} \left[ (-15 - 15k + 5k^2 + 25k^3 + 9k^4)c_\phi - 15l^2(1 + k - 3k^2 + k^3 + k^4)c_2 + 15k^2l^2(-2 + 2k + k^2)c_1 \right], \]

\[ \lambda_6 = \frac{1}{1680k^5(1+k)l^7} \left[ (-105 - 105k - 35k^2 - 35k^3 + 77k^4 + 203k^5 + 75k^6)c_\phi - 105l^2(1 + k - k^2 - k^3 - k^4 + k^5 + k^6)c_2 + 105k^2l^2(1 + k - 2k^3 - k^4)c_1 \right]. \]
Pin-jointed bar with linear prebuckling paths

\[ \lambda_2 - \lambda_4 \text{ curves (} a_1 = 0 \text{):} \]

\[ T = Q \]

\[ T \]

\[ Q \]

\[ \lambda_4 \]

\[ \lambda_2 \]

\[ \lambda_4 \]

\[ \lambda_2 \]
Conclusions

- continuous increase of thickness
  ✔ no conversion into imperfection insensitivity

- decrease of the initial rise
  ⇒ transition to situation with no loss of stability
  (with reduction of stability limit)

- increase of the stiffness of spring
  ⇄ conversion from imperfection sensitivity into insensitivity
  (depending on thickness)